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Robust Shortest Path Problem: Models and Solution Algorithms

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By

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DISSRTATION

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ABSTRACT

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Shortest path is a key component of several network related problems and has been widely used and applied in numerous disciplines such as transportation, logistics and telecommunication networks. This problem in the base deterministic settings lends itself to elegant and efficient solution methods. Nevertheless, the initial formulation is limited in number of ways; one such limitation is the need to accounting for inherent uncertainty in real world transportation networks. In recent years there has been a growing interest in incorporating uncertainty within the transportation network analysis models and particularly the shortest path problem. This dissertation contributes to the growing body of literature in dealing with uncertainty in shortest path problem by developing formulations as well as efficient solution methodologies for these class of problems.

There are number of approaches across the literature for incorporating the stochastic features of network related parameters such as travel time into the shortest path problem. One such approach is to minimize mean-risk analysis which is to minimize both the average cost and the risks arising from the uncertainty assumptions. The chief complication of such modeling approach is that the size of the nonlinear part of the objective function will increase for the large size real world network problem which undermines the efficiency of the existing solution approach. In response to such needs a solution methodology based on outer approximation (OA) strategy is proposed and customized which is highly efficient for real world large size instances.

In addition, in this dissertation a robust optimization approach for the shortest path problem where travel cost is uncertain and exact information on the distribution function is unavailable has been applied. Robust shortest path under such conditions is shown to be formulated as a binary nonlinear integer program, which can then be reformulated as a mixed integer conic quadratic program.

Finally, both two modeling frameworks provide a generalization in which links have two cost components, representing the expected cost and risk measure on the links – the former term is additive, but the latter is not. In the third and final part of this dissertation, a solution methodology for general formulation of shortest path problem with non-additive continuous convex travel cost functions is presented.

Dedication:

To my parents, Mehry and Houshang, to my grandmom, Vahdat, and to my brothers, Farhad and Farzad, without whom none of my success would have been possible

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Chapter 1 Introduction

Transportation systems are a critical component of today's modern society contributing to its economic vitality and growth. The primary role of such systems is the efficient movement of commodities (passengers or goods) from the origins to destinations while ensuring safety and reliability. Typically, the transportation system comprises of all modes of transportation (Mass Transit, Highway, Freight, Rail, and Aviation) that moves millions of passengers and millions of tons of goods between origins and destinations. The operation of these systems is taking place in various forms of network structures where system users such as urban passengers or freight companies are seeking for the best routes to travel from origins to destinations.

Providing the best routing strategy for the users of the transportation systems has been the center of attention for many years and owing to its broad applications such problems have been intensively studied by transportation researchers and practitioners. However, with recent challenges in urbanization, ever growing demand for transportation services, slow development of city infrastructure coupled with arrival of new technologies such as telecommunication and information technologies, finding the most desired routing policy is by no means an easy task. Faced with such complexities, more and more attention is being paid to the investigation of better models capable of describing user's behaviors in selecting routes which play a crucial role in determining the network conditions and can significantly alter other long term planning decisions such as network design.

The shortest path problem is a key component of many traffic routing decision frameworks. Shortest path problem due to its broad applications in networks related models have attracted much attention from researchers in numerous fields, ranging from communication to transportation. Traditionally in transportation networks, shortest path problems involve finding the paths with minimum cost. The primary assumption in such a modeling framework is that the link travel times or costs are known and deterministic. However, this assumption may not always be true as the link travel times especially in urban road networks are highly uncertain due to events such as accidents, bad weather, traffic demand fluctuations, physical bottlenecks, and capacity degradations. Consequently, there is a growing paradigm shift by the researchers from the modeling standpoint towards incorporating different features such as travel time reliability or delay minimization to account for the prevailing uncertain conditions within the transportation network.

There are number of approaches across the literature for incorporating the stochastic features of network related parameters such as travel time into the shortest path problem. One such approach is to minimize mean-risk analysis proposed by Markowitz [1987] which is to minimize both the average cost and the risks arising from the uncertainty assumptions. One way of capturing the risk minimization in shortest path problems is to determine the path which minimizes a linear combination of average costs and the standard deviation. In such cases the data needed for the correlations can be measured and calibrated through the data received by technologies such as ATIS [Seshadri and Srinivasan 2012, Sen et al. 2001]. In this framework standard deviation is a measure of unreliability and is the risk associated with travel time variations and the goal is to find a path with minimum tradeoff between unreliability and travel cost. This type of modeling is sometimes referred to mean-variance or mean-standard deviation shortest path problem in the literature.

Nevertheless, a major drawback of this approach is that it is sometimes difficult to accurately define the moments of the probability distribution associated with the uncertain links cost and thus the variance-covariance matrix for the network cannot be determined. There can be errors in the parameters estimation as there are many factors such as traffic conditions, weather, and accidents whose effects on the travel time are hard to quantify. There are multiple directions of research in the literature in order to deal with such issues. One popular method is to consider that the link travel costs are uncertain and belong to a bounded interval which represents a range of possible values for the link cost variation. The goal of this modeling approach is to find a path which is immunized against the worst case outcome of the uncertain travel time. This approach provides solutions which are "robust" against the existing bounded uncertainty. Following such an approach, the robust shortest path is defined as the path that minimizes the maximum path cost over all the realizations of the links cost belongs to the interval. In other words, the robust path defined in here corresponds to the path with the best worst case cost. However, this type of modeling may lead to over conservative solutions. In response to this need in the literature, robust optimization techniques have been developed by the researchers which provide an adjustable framework in order to avoid over-conservativeness of the solution and target a desired level of robustness for dealing with the uncertainty.

Both the modeling frameworks discussed earlier provide a generalization in which links have two cost components, representing the expected cost and risk measure on the links – the former term is additive, but the latter is not. The chief complication of such modeling framework is that the Bellman principle of optimality is not valid and thus classical label setting and label correcting algorithms are no longer applicable. This give rise to a class of programs which are collectively called non-additive shortest path problem.

The focus of this dissertation is to provide a comprehensive treatment of the robust shortest path in transportation networks, in terms of modeling approach as well as the solution methodology. Different types of uncertainty and modeling assumptions along with their effect on the routing decisions have been discussed and presented. Furthermore, a general formulation of the non-additive shortest path with convex attribute functions has been proposed and analytically solved. The solution methodology developed in here is particularly attractive as it provides the global optimal solution for this class of programs and computationally efficient for large size real world network problems

1.1 Motivation Examples

In this section, simple motivating examples presenting different models of the robust shortest path along with the example for the general convex non-additive shortest path problem have been presented to motivate the research conducted in this dissertation. In particular, the first example deals with robust shortest path with available variance-covariance matrix. The second example emphasizes on the robust shortest path with bounded interval cost for every link and finally the third example provides a case with non-additive attribute functions. In all the three examples the focus is toward presenting the difference of the classical shortest path problem with the existing approaches in finding the routing decisions. Figure 1.1 presents the example network with basic deterministic costs.

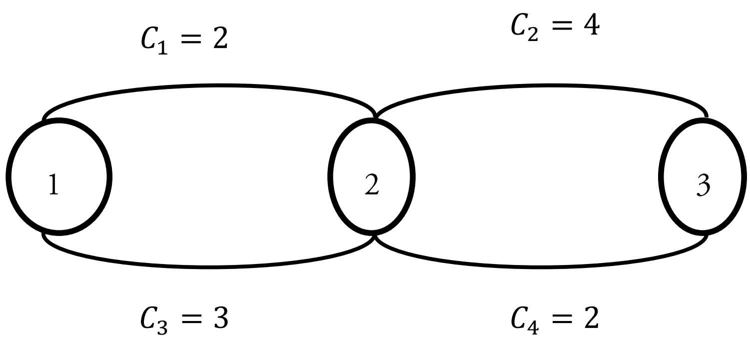


Figure 1-1: Example Network

1.1.1 Motivation example 1: Mean-standard deviation shortest path

The purpose of this illustrative example is to account for the shortest path decision in the presence of the variance-covariance matrix for the link travel cost and the comparison with the classical shortest path problem. Towards this end, consider a single origin-destination pair from node 1 to node 3 with the following variance-covariance matrix:

$$
\begin{array}{c|cccc|}
 & a_1 & a_2 & a_3 & a_4 \\
a_1 & 4 & 1.2 & 0.875 & 0.75 \\
a_2 & 1.2 & 3 & -0.75 & 0.5 \\
a_3 & 0.875 & -0.75 & 2 & 1 \\
a_4 & 0.75 & 0.5 & 1 & 3
\end{array}
$$

Figure 1-2: Covariance Matrix for the example network

First, assuming the classical shortest path problem with no standard deviation consideration the shortest part decisions can be simply calculated by finding the path with minimum cost connecting origin node 1 to destination node 3. Consequently, in this case the shortest path is comprised of the links 1 and 4 with the total cost of 4. Now, given the availability of the variance-covariance matrix the goal is to find the path with minimum both travel cost as well as the unreliability measure defined here as the standard deviation of the travel time of the path. Table 1.1 provides the cost as well as the standard deviation associated to every path between origin-destination pair (1, 3):

Table 1-1: Mean+Std cost for all the paths of example network

Path (Link Numbers)	Mean Cost	Standard Deviation	Mean+STD Cost
1-2	6	2.72	8.72
1-4	4	2.82	6.83
3-2	7	2.34	9.35
3-4	5	1.73	6.73

As one may observe from the results, the optimal path with no standard deviation consideration is the path comprised of links 1 and 4, whereas when considering the standard deviation as a measure travel time variations into the shortest path calculation this path is no longer the optimal path and the path comprised of links 3 and 4 tends to have a total lower cost and thus is the optimal shortest path decision. This small illustrative example confirms the fact that accounting for travel time variations in terms of added standard deviation into the classical shortest path problem may essentially lead to a different set of decisions.

1.1.2 Motivation example 2: Shortest path with bounded interval link cost

The focus of this example is to demonstrate the shortest path problem in the face of uncertain link travel cost where the cost belongs to a bounded interval. Consider the same network used in the previous section with 3 nodes and 4 links where links travel costs are expressed through bounded intervals presented in Figure 1-3.

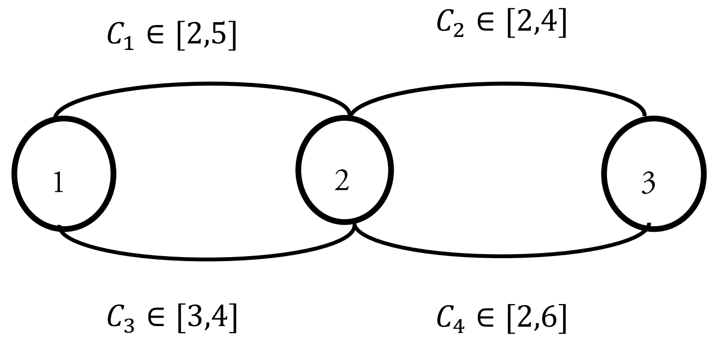


Figure 1-3: Example network with bounded interval travel cost

The numbers presented in the interval represent the minimum and the maximum of travel time for every link. One possible solution is to consider the worst case scenario for the cost of every link. This means that the cost for every link corresponds to the higher bound of the interval. Therefore, in the case of small illustrative network shown above, the shortest path which connects origin node 1 to designation node 3 is comprised of links 3 and 2 with total cost of 8. Nevertheless, considering the cost at its worst case can result to very conservative solutions. In response to this need, robust optimization strategy has been proposed which provides a framework where the decision maker can adjust the solution based on a desired level of protection. This example although simple, motivates the need for a modeling framework.

1.1.3 Motivation example 3: Shortest path with convex non-additive attribute function

The focus of the third example is to provide a case for the non-additive shortest path problem where the decision maker is seeking to minimize the tradeoff between two nonlinear attribute functions. More specifically, in this example we consider two nonlinear convex attribute functions and show how accounting for nonlinear attributes for travel time would lead to decisions different from the classical approach. Let's consider two non-additive convex functions $v(\alpha) = \frac{\alpha^2}{100}$ and $v(\alpha) = 100e^{-\frac{\alpha}{2}}$ where α is the cost of the path. The results in terms of the cost for every path of the test network in Figure 1-1 and the total cost of the path are presented in Table 1-2.

Table 1-2: Non-additive Path cost for all the paths of example network

Path (link numbers)	First attribute function	Second attribute Function	Classical SP Cost	Total Cost
1-2	0.36	4.98	6	11.34
1-4	0.16	13.53	4	17.69
3-2	0.49	3.02	7	10.51
3-4	0.25	8.21	5	13.46

The results presented in the above Table show that the path cost based on the non-additive cost function attributes as well as the results of the classical shortest path. The results clearly show that the shortest path with non-additive attribute cost function is different from the classical setting. More specifically, given the two non-additive attribute functions the shortest path is comprised of links 3 and 2 with the total cost of 10.51 while based on the classical setting the shortest path lies on links 1 and 4. This example again confirms that accounting for different attributes besides the average cost of travel can lead to different shortest path solutions.

1.2 Dissertation Contributions

The contributions of this dissertation are the following:

- Study the mean-standard deviation formulation for the robust shortest path while considering the availability of the correlation matrix for the links travel time. Several other researchers have adopted the mean=standard deviation formulation for the robust shortest path problem but with restrictive assumptions on the correlation matrix and network sizes [see for e.g.: Xing and Zhou 2010, Sen et al 2001, Sivakumar and Batta 1994]. However, the correlation matrix poses significant computational burden on the problem for real size networks as the model will become highly nonlinear making many of the available methods in the literature intractable. In response to such a need in the literature a solution methodology based on the outer approximation algorithm has been developed and implemented which is independent of the structure of the covariance matrix and its efficiency cannot be undermined by increase in the size of the correlation matrix. Furthermore, the proposed solution methodology theoretically provides the global optimal solution for this class of problem.
- Development of a robust modeling formulation for the shortest path problem while assuming limited information on the distribution of links travel time. The model provides adjustable shortest path decisions while travel time uncertainty is bounded and belongs to an ellipsoidal uncertainty set.

- Development and solve of a general formulation for the non-additive shortest path problem with convex nonlinear attribute function. The computational complexities of the model have been presented and the formulation has been tested and examined with several attribute functions on varying real world test problems.
- A solution methodology based on the outer approximation has been developed and its computational features have been derived and specifically studied for solving all three suggested formulations. Furthermore, given that convexity and differentiability of nonlinear functions the outer approximation algorithm is proved to converge to global optimality. Ultimately, through extensive numerical experiments the efficiency and computational advantages of the proposed framework is tested on large sized test instances and the results are compared with the state-of-the art software's.

1.3 Outline of This Dissertation

The remainder of this dissertation is organized into 5 sections:

- *Chapter 2: Literature Review*

Chapter 2 provides a thorough literature review of three different streams of existing literature: stochastic and robust shortest path problem, robust optimization, a general overview of the outer approximation algorithm, that provide an overall view of the concepts that will be utilized in the subsequent chapters of this dissertation.

- *Chapter 3: Mean-Standard Deviation Shortest Path Problem*

Chapter 3 sets forth the general formulation for the mean-standard deviation shortest path problem while considering full correlation for links travel time of the networks. The analytical mathematical programming formulation and the corresponding solution algorithm are presented. The performance of the solution is also compared with state of the art solvers and the methodology presented by Sen et al. [2001] for solving this class of programs

- *Chapter 4: Robust Shortest Path Problem with uncertain Links Travel Time Distribution Function*

In Chapter 4 the robust optimization approach in providing an analytical formulation for the robust shortest path problem while assuming ellipsoidal uncertainty set is applied. The robust optimization assumptions along with the model for the robust shortest path problem and the associated computational results on various increasing size networks are provided.

- *Chapter 5: Non-additive Shortest Path Problem*

Chapter 5 provides the formulation for the non-additive shortest path problem with convex non-additive function. The associated algorithm for solving this class of programs is presented and results are discussed based on different forms of the travel cost attribute functions on varying size real world networks.

- *Chapter 6: Conclusion and Future Recommendations*

Finally Chapter 6 provides summary and some concluding remarks for each application as well as some future research directions regarding the proposed algorithmic framework.

Chapter 2 Literature Review

The shortest path problem is a central problem in the area of operations research which has been extensively studied in various forms. The traditional shortest path formulation lends itself to elegant and efficient solution methods, which makes it a useful basis for more complex routing and scheduling problems. However, despite significant amount of research being conducted on the study of the shortest path problem with travel time uncertainty, there is still scope for further modeling approaches as well as solution methodologies to better model and solve these types of problems based on the existing conditions in real practice. This chapter tries to provide the motivations of this dissertation with respect to existing literature both in terms of the modeling approach and the solution methodology. This chapter starts by reviewing a number of previous works in the area of stochastic and robust shortest path problems with features such as reliability, on time arrival etc., followed by an overview on robust optimization as one of the modeling approaches, and outer approximation algorithm as the solution methodology.

2.1 Classical Shortest Path Problem

The shortest path problem in its base deterministic form is to find a path in a weighted graph with the lowest possible cost. The shortest path problem is among the most studied problems in the area of operations research, computer science, and transportation engineering, and often serves as the subproblem for many optimization related problems. Efficient algorithms have been developed and extended over the past 50 years for solving the shortest path problem which can be categorized under two big groups of label-setting and label-correcting [Bellman, 1958; Dijkstra, 1959; Dreyfus, 1969]. A comprehensive review of the classical shortest path algorithms, their extensions, and performance can be found in Ahuja et al. [1994] and Cherkassky et al. [1996] to only name a few.

Consider a directed transportation network *G(V,E)* in which *V* is the set of nodes and *E* is the set of links. The mathematical programming formulation for the shortest path problem can be written as follows:

$$Z = min \sum_{\substack{i,j \in E \\ (r,s)}} C_{ij} x_{ij}^{rs} \qquad (2\text{-}1)$$

$$\sum_{(i,j) \in E} x_{ij}^{rs} - \sum_{(j,i) \in E} x_{ij}^{rs} = \begin{cases} 1 & if\ i \in r \\ 0 & otherwise \\ -1 & if\ i \in s \end{cases} \qquad (2\text{-}2)$$

$$x_{ij}^{rs} = \{0\ or 1\}, \quad \forall a_{ij} \in E \qquad (2\text{-}3)$$

In the above formulation, $a_{ij} \in E$ represents a link in the network between node *i* and *j*. Let $r \in V$ represent the set of origins and $s \in V$ the set of destination nodes. Therefore *(r-s)* denotes an origin destination pair. Let C_{ij} represent the link travel cost. In the above formulation, x_{ij}^{rs} is a binary decision variable which determines whether link a_{ij} lies on the shortest path for origin destination pair *(r-s)* or not. The above formulation is seeking to determine a shortest path with minimum cost from origin node r to the destination node s while the flow balance is enforced through the constraints (2-2). More specifically, constraints (2-2) ensure that the flow into a node is equal to the flow out of that node for all the nodes in the network ensuring that the solution has a path structure.

The classical shortest path problem may not always provide the best possible solution for real world traffic applications due to the prevailing stochastic conditions in transportation networks. Thus, different approaches and consequently solution techniques are required to tackle the uncertainty conditions affecting these networks. This gives rise to the stochastic shortest path problem where travel time or the network conditions can be uncertain. The next section reviews the relevant literature of the stochastic shortest path problems.

2.1.1 Stochastic Shortest Path Problem (SSPP)

The stochastic shortest path problem (SSPP) assume that the link travel times are random or network conditions are subject to variations or even failure. In particular, stochastic models for the shortest path problems are intended to capture the underlying uncertainty in the routing decisions while accounting for random fluctuations in network related parameters such as links cost or in the structure of the network, such as nodes and links failure. Over the years, different versions of the problem have been extensively studied and extended in various aspects throughout the literature.

Hall [1986] studied the stochastic shortest path problem while assuming random time-dependent link travel times. The stochastic shortest path problem was solved using a dynamic programming based algorithm and the results were demonstrated on a transit based network structure. However, if link costs do not vary with time, the shortest expected-cost path can be found by replacing each link's cost with its expected cost and applying a deterministic shortest path algorithm; this problem has been analyzed in depth by Frank [1969], Mirchandani [1976], and Sigal et al. [1980]. When link costs are time-dependent, Bellman's principle fails and the worst-case computation time for an exact solution is exponential [Hall, 1986; Fu and Rilett, 1988]. An exact solution method based on dominance arguments have been developed by Miller-Hooks and Mahmassani [2000]. Least expected path travel cost was also investigated under online information provision [Polychronopoulos and Tsitsiklis, 1996, Cheung, 1998, Waller and Ziliaskopoulos, 2002 and Fan et al., 2005].

Researchers have explored many ways of incorporating the importance of reliability factors into shortest problems. Several studies in the transportation domain show the importance of reliability in affecting travel choice decisions. A diversity of reliability measures have been used: Pinjari and Bhat [2006] used the maximum additional time which could be needed, compared to the average time; Small et al. [2005] and Liu et al. [2004] used the difference between the 80th and 50th percentile travel times (roughly one standard deviation in many probability distributions), and da Palma and Picard [2005] evaluated several competing specifications based on standard deviation, variance, constant relative risk aversion, and constant absolute risk aversion. All of these studies found that the reliability measure was highly significant, and often comparable to the importance of average travel time in the route choice decision. A more thorough review of theoretical and empirical research in this area can be found in Bates et al. [2001] and Noland and Polak [2002]. Another general approach is the introduction of a (possibly nonlinear) disutility function representing the weight of a path as a function of arrival time, and seeking a path minimizing expected disutility. This approach was pioneered by Loui [1983] and Eiger et al. [1985] for the cases of linear and exponential disutility functions, and by Murthy and Sarkar [1996] for the case of quadratic functions. Loui [1983] also explored an approach based on stochastic dominance, a technique also used by Miller-Hooks and Mahmassani [2003]. Nie, Wu, and Homem-de-Mello [2012] applied higher-order stochastic dominance techniques to model risk aversion.

A somewhat different approach is to find a path or policy maximizing the probability of arriving at the destination by a specified threshold time, an approach known as the arriving-on-time problem. This variant has been investigated by Fan et al. [2005], Fan and Nie [2006], Nie and Fan [2006], Nie and Wu [2009] and Chen et al. [2013].

2.1.2 Mean-Standard Deviation Approach

Another approach is to incorporate travel time variance or standard deviation directly into the formulation, either as a constraint [Sivakumar and Batta, 1994], as the sole objective [Gao, 2005], as an additional objective in addition to minimizing expected travel time [Sen et al., 2001; Xing and Zhou, 2011], or as an additional term in the objective function [Hutson and Shier, 2009]. Boyles and Waller [2011] applied the latter formulation to the stochastic minimum cost flow problem, a generalization of the stochastic shortest path problem. It is important to note that, in all of the mentioned works the mean and standard deviation variance of link travel time is assumed to be available or can be calculated. For instance Fu and Rilett [1998] provided an approximation algorithm by which mean and variance of the travel time can be estimated. Ashok and Ben-Akiva, [1993], Zhou and Mahmassani [2007] proposed a travel time variance estimation method a recursive estimation error propagation formula. The mean-standard deviation approach is behaviorally intuitive and easiest to calibrate, and thus is adopted in this dissertation.

2.2 Stochastic Shortest path Problem-Robust Optimization Approach

This section briefly reviews the robust optimization literature and its shortest path applications.

2.2.1 Robust Optimization

Generally, there are two streams of research for dealing with the uncertainty in mathematical programming- stochastic programming and robust optimization. Stochastic programming relies on the probability distribution of the uncertain parameters while robust optimization model the uncertain variables using uncertainty sets rather than probability distributions. The pioneering work for providing the robust solution for mathematical program is by Soyster [1973] who built a formulation capable of providing feasible solution for a linear optimization model with the input data from an interval. However, it took more than two decades until the concept of robust optimization based on the minimization of the worst case objective concept was introduced and comprehensively studied for different types of mathematical programs through the independent seminal works of Ben-Tal and Nemirovski [1998,1999,2000,2001], and El Ghaoui et al. [1998,1999]. In their works, limited assumptions such as availability of bounds are made on the uncertain possible values, and a minmax problem is solved which provide the worst case realization of the objective functions. They also introduced the notion of the uncertainty set which is a mechanism to adjust the conservativeness of the solution. Assuming an ellipsoidal uncertainty set, they provided the robust formulation for various classes of mathematical programs such as linear, second order cones and convex programs. Bertismas and Sim [2004] developed the concept of polyhedral uncertainty set and showed that the robust formulation for a linear programs will remain in the realm of linear programming. Thus the formulation for the robust model is related to the assumptions made on the shape of the uncertainty set. Bertismas and Sim [2004] also provided the robust counterpart of linear programs and studied the feasibility of the robust solution with respect to the desired level of protection given. Over the past decade researchers have extended the robust optimization concept in many directions and this approach has been incorporated in various fields such as inventory control [Bertsimas and Thiele, 2006], location problem [Gülpinar, et al. 2013] and so forth. For further information regarding the robust optimization strategy and its application the readers are referred to Bertsimas et al. [2011].

2.2.2 Robust optimization for the shortest path problem with uncertain travel time distribution

Robust optimization in the context of shortest path and other network flow problems has been applied by many researchers. There are two major assumptions on the robust optimization strategy which eventually lead to two different modeling formulations for the robust shortest path problem across the literature. The first type of the models consider bounded interval uncertainty sets while in the second approach uncertainty is described through finite discrete scenarios. The robust shortest path based on the interval uncertainty can be found in the works of Karasan et al. [2001], Kasperski and Zielinski [2006], Montemani and Gambardella [2004 and 2005] and Kwon et al. [2013] while Gabrel and Murat [2010a], Kouvelis and Yu [1997], Yu and Yang [1998] studied the robust shortest path with discrete uncertainty sets. Bertsimas and Sim [2003] developed a polynomial-time approach based on the polyhedral uncertainty set (in contrast to alternate formulations of more general robust discrete optimization problems, which can lead to NP-hard problems, [Kouvelis and Yu, 1997]. Furthermore, robust optimization approaches have been applied to network flow problems, readers may refer to Bertsimas and Sim [2003], and Atamturk and Zhang [2007] for further information. However, none of the available works in the literature considered the robust shortest path problem with ellipsoidal uncertainty sets while assuming the availability of bounds on the uncertain parameters. The assumption of the ellipsoidal uncertainty set in construction of the robust formulation although will change the formulation to a nonlinear program model but will provide less conservative solutions as discussed by Chen et al [2007].

2.2.3 Non-additive Shortest Path Problem

This section reviews relevant literature on non-additive shortest path problem. Mirchandani and Soroush [1985] proposed an exponential algorithm for general non-increasing quadratic utility function. Henig [1986] investigated the shortest path problem with quasiconcave or a quasiconvex disutility functions and developed an approximation scheme with applying a label correcting algorithm to obtain the optimal path decision. Loui [1983] and Eiger et al. [1985] show that for linear and exponential utility Bellman principle holds and thus dynamic programming is still applicable.

Several studies focused on providing a general formulation for the non-additive shortest path problem along with the solution methodology. Scott and Bernstein [1997] considered a non-linear and non-decreasing disutility function based on travel time and tolls and presented an iterative pruning procedure by which the optimal path is selected through evaluating the cost of the set of pareto optimal paths. Gabriel and Bernstein [2000] suggest a heuristic method for solving the non-additive shortest path with general continuous cost function. Tsaggouris and Zaroliagis [2004] developed an exact algorithm with combining Lagrangian relaxation and extended hull approach to solve the shortest path problem for two non-additive monotone convex functions with resource constraint. Murthy and Sarkar [1997] embedded a relaxation based pruning technique based on the principle of fathoming nodes in a branch and bound process in a label setting framework to determine the shortest path for concave quadratic disutility functions. Chen and Nie [2013] considered a bicriterion shortest path with nonlinear utility function and proposed a sequential piecewise linearization technique in order to solve the problem. Furthermore, non-additive assumption on link travel cost has also been considered in traffic assignment framework [Gabriel, and Bernstein, 1997a; Gabriel, and Bernstein, 1997b; Lo, and Chen, 2000, and Chen, et al., 2010].

2.2.4 Outer Approximation Algorithm

This section describes an outer approximation (OA) algorithm designed to solve the MICQP formulation of the robust shortest path problem. OA is a classical method based on the cutting plane algorithm proposed by Duran and Grossman [1986] for solving convex mixed-integer nonlinear programs (MINLPs). OA works by decomposing the original MICQP into a sub-problem (SP) which is a non-linear program (NLP) and a master problem which is a mixed integer program (MIP). The NLP subproblem is obtained by fixing the integer variables in the original MINLP. The solution of nonlinear subproblem is feasible for the MINLP and therefore provides upper bounds for the original problem. Moreover, the OA master problem (MP) is an equivalent linear representation of the original MINLP and gives the lower bound solution for problem. According to Bonami et al. [2009] the OA algorithm attains the solution of the original MINLP after a finite sequence of solving SP and MP if the original problem is convex, differentiable and certain constraint qualification condition is satisfied. A comprehensive literature of OA can be found in Duran and Grossman [1986] and Fletcher and Leyffer [1994]; below, we provide a brief overview of this method.

Consider the general MINLP where $f(x, y)$ and $g(x, y)$ are both convex differentiable function and X and Y are sets of discrete and continuous variable respectively.

$$\text{Min } f(x, y) \tag{2-4}$$

$$g_i(x, y) \leq 0 \qquad \forall i = 1, .., I \tag{2-5}$$

$$y \in Y \text{ and } x \in X \tag{2-6}$$

The outer approximation subproblem is solved by fixing the integer variables from the original MINLP at every iteration h, yielding a nonlinear subproblem (SP). The algorithm requires initial integer feasible solution at the first iteration, which can be difficult in general. However, for our mean-standard deviation shortest path problem a feasible integer solution can easily be obtained by solving a classical shortest path problem. Using x^h to denote the integer solution at iteration h, the OA subproblem is:

SP:

$$\text{Min } f(x^h, y) \tag{2-7}$$

$$g_i(x^h, y) \leq 0 \qquad \forall i = 1, \dots, I \tag{2-8}$$

$$y \in Y \tag{2-9}$$

Let y^h denote the optimal solution to SP. Assuming that the Karush-Kuhn-Tucker conditions hold at (x^h, y^h), the convexity properties of $f(x, y)$ and $g(x, y)$ imply that inequalities (2-11) and

(2-12) are valid linear approximations for the nonlinear terms of the original problem. The inequalities produced at each iteration are collected to construct the OA master problem. The resulting master problem (MP) is a mixed integer linear program. The generic form of the OA MP is:

OA-MP:

$$\text{Min } \eta \tag{2-10}$$

$$f(x^h, y^h) + \nabla f(x^h, y^h)^{\mathrm{t}} \begin{pmatrix} x - x^h \\ y - y^h \end{pmatrix} \le \eta \qquad \forall h = 1, \dots, H \tag{2-11}$$

$$g_i(x^h, y^h) + \nabla g_i(x^h, y^h)^{\mathrm{t}} \begin{pmatrix} x - x^h \\ y - y^h \end{pmatrix} \le 0 \qquad \forall h = 1, \dots, H, \; i = 1, \dots, I \tag{2-12}$$

$$y \in Y \text{ and } x \in X \tag{2-13}$$

As mentioned before, the master problem is augmented at each iteration h with a set of constraints until the updated upper bound and the updated lower bound for the OA algorithm converge. The convergence criterion in the outer approximation algorithm is defined as the absolute or relative difference between the updated solution of the subproblem and master problem. The iteration converges and the algorithm will stop if the difference between the updated lower bound and the updated upper bound is less than a certain predefined threshold.

This chapter provided an overview of literature of different variants of the robust shortest path relevant to the models considered in this dissertation. Despite significant amount of work in the area of the robust shortest path problem, there is still scope for further developing the modeling formulations as well as the solution methodology for dealing with this class of programs. Furthermore, there is still limited amount of which can theoretically provide the global optimal solution for this type of problems. The next chapter focuses on the mean standard deviation shortest path problem.

Chapter 3 Mean-Standard Deviation Shortest Path Problem

3.1 Introduction

Finding shortest paths in a network is a central problem in routing and logistics modeling, telecommunications, and many other domains. In its most basic form, the cost of a path is the sum of the costs on its component links, a separability property which produces extremely efficient solution methods. This chapter addresses the robust shortest path problem, a generalization in which links have two cost components, representing the expected cost and cost standard deviation on the links – the former term is additive, but the latter is not. In particular, we introduce a novel solution method for this problem based on reformulation as a conic program.

The need to account for uncertainty in routing problems is well-known. Link travel times or costs are almost always stochastic. In the logistics domain, this is often due to congestion effects, poor weather, or capacity disruptions from incidents. When costs are uncertain, the problem of finding a "minimum-cost path" is ill-posed, since the cost of paths cannot be known in advance. The simplest adaptation is to find the least expected-cost path, but this ignores the issue of reliability raised by cost stochasticity. In many applications, particularly involving supply chains, an ideal path not only has low expected cost or travel time, but also low variance or standard deviation – being able to accurately forecast the travel time or cost is often just as important as minimizing the mean value. By finding a path which minimizes a weighted sum of mean travel time and travel time standard deviation, both of these issues are addressed. Furthermore, by controlling these weights, decision makers can find the right balance between these factors based on their risk tolerance and attitudes.

As discussed in the literature review, many researchers have formulated and solved variations of this problem. Thus a weighted sum of mean and standard deviation is the most intuitive and useful formulation; for instance, while minimizing a weighted sum of mean travel time and travel time *variance* preserves additivity, the units are incommensurate, making it harder to calibrate and interpret the model. Broadly speaking, solution methods for the mean-standard deviation shortest path problem can be classified as network algorithms or as mixed-integer programs. In this chapter, a novel conic programming formulation is suggested, and also an outer approximation algorithm is developed for this formulation of the problem, and its performance is shown to be extremely competitive.

Next section introduces the conic programming reformulation, followed by a description of the corresponding outer-approximation algorithm. Computational performance of the algorithm are discussed in the final section of this chapter.

3.2 Problem Statement

Let *G(V,E)* be a graph where *V* is the set of nodes and *E* is the set of directed links, with elements a_{ij} denoting the links connecting node i to node *j*. We formulate the shortest path problem from one origin r to all destinations $s \in S \subseteq V$, where *S* is the set of destination nodes. Let x_{ij}^{rs} represent the binary decision variable equal to one if and only if link a_{ij} lies on the chosen path for origin destination (OD) pair (*r,s*). In this section, we assume that link travel cost is a random variable $\tilde{C}_{ij}$ with a known mean c_{ij} and variance σ_{ij}. Let $cov(a_{ij}, a_{lk})$ denote the travel cost covariance between links a_{ij} and a_{lk}. We only require two assumptions on the specific probability distributions and correlation structure: (1) the existence of at least one path with finite variance; and (2) that the standard deviation of any path containing a cycle is greater than the standard deviation of that same path with the cycle removed. The latter assumption is common in the stochastic shortest path literature [see, for instance, Sen et al., 2001] and we do not believe it to be restrictive: in most practical applications, we find it unlikely that travel time variability can be decreased by traversing a cycle, and that such possibilities only arise with covariance matrices that do not characterize field conditions.

We seek the path which minimizes a linear combination of expected and standard deviation of travel costs. The resulting binary integer nonlinear programing (BINLP) formulation P1 for one origin r to all destinations $s \in S$ is given below:

P1:

$$Min\ w \sum_{\substack{a_{ij} \in E \\ s \in S}} c_{ij} x_{ij}^{rs} + (1 + w) \sum_{s \in S} \sqrt{\sum_{a_{ij} \in E} \sigma_{ij}^2 {x_{ij}^{rs}}^2 + \sum_{a_{ij} \in E} \sum_{\substack{a_{lk} \in E \\ l,k \neq i,j}} cov(a_{ij}, a_{lk}) x_{ij}^{rs} x_{lk}^{rs}} \qquad (3\text{-}1)$$

$$\sum_{j: a_{ij} \in E} x_{ij}^{rs} - \sum_{j: a_{ji} \in E} x_{ji}^{rs} = \begin{cases} 1 & \forall\, i = r \\ 0 & \forall\, i \neq r, i \neq s \\ -1 & \forall\, i = s \end{cases} \quad \forall s \in S \qquad (3\text{-}2)$$

$$x_{ij}^{rs} \in \{0,1\} \qquad \forall a_{ij} E \quad \forall s \in S \qquad (3\text{-}3)$$

In the above formulation $w \in [0,1]$ is the reliability coefficient adjusting the importance of cost standard deviation relative to mean cost, as expressed in the objective function (3-1). Equation (3-2) is the flow balance constraints for each origin-destination pair. Also, note that our goal is to determine the robust shortest paths from one origin r to all destinations $s \in S$. Equation (3-3) enforces the binary constraint on the decision variable. In order to further exploit the properties of the above BINLP, we reformulate P1 as mixed integer conic quadratic program (MICQP) by introducing a positive variable $t^{rs} \geq 0$, and using the fact that $x_{ij}^{rs} = x_{ij}^{rs\,2}$. Note that the reformulated model P2 has a linear objective function, and that the constraints (3-6) are quadratic.

P2:

$$Z = Min\ w \sum_{\substack{a_{ij} \in E \\ s \in S}} c_{ij} x_{ij}^{rs} + (1-w) \sum_{s \in S} t^{rs} \tag{3-4}$$

$$\sum_{j:a_{ij} \in E} x_{ij}^{rs} - \sum_{j:a_{ji} \in E} x_{ji}^{rs} = \begin{cases} 1 & \forall\, i = r \\ 0 & \forall\, i \neq r, i \neq s \\ -1 & \forall\, i = s \end{cases} \quad \forall s \in S \tag{3-5}$$

$$\sqrt{\sum_{a_{ij} \in E} \sigma_{ij}^2 x_{ij}^{rs\,2} + \sum_{a_{ij} \in E} \sum_{\substack{a_{lk} \in E \\ l,k \neq i,j}} cov\left(a_{ij}, a_{lk}\right) x_{ij}^{rs} x_{lk}^{rs}} \leq t^{rs} \quad \forall s \in S \tag{3-6}$$

$$x_{ij}^{rs} \in \{0,1\}, t^{rs} \geq 0 \quad \forall a_{ij} \in E,\ \forall\, s \in S \tag{3-7}$$

The proposed formulation (P2) for the robust shortest path problem belongs to a class of mixed integer nonlinear program known as mixed integer conic quadratic program (MICQP). If the binary constraints are relaxed, the above formulation becomes a conic quadratic program (CQP), also known as a second order cone program (SOCP). This class of convex optimization problems has attracted the attention of many researchers. CQP are computationally tractable due to their special structure and can be solved by polynomial time interior point algorithms. The overview of the CQP is provided in Ben-Tal and Nemirovski [2001], Alizadeh and Goldfarb [2003] and Lobo et. al [1998]. One potential way to solve the MICQP is through branch and bound algorithms in which a relaxed continuous conic quadratic program (QCP) is solved at the nodes of tree.

In the above formulation, equation (3-6) is the conic quadratic constraint which fits into the general assumption for conic programs. Furthermore, constraint (3-6) can be simplified as follows:

$$\sum_{a_{ij} \in E} \sum_{a_{lk} \in E} x_{ij}^{rs} \, cov(a_{ij}, a_{lk}) \, x_{lk}^{rs} \le t^{rs2} \qquad \forall s \in S \tag{3-8}$$

The transformation of model BINLP (P1) into the MICQP (P2) facilitates solution through embedded branch and cut algorithm in solvers like CPLEX and MOSEK. However, the covariance matrix grows quadratically with network size, imposing computational burdens and undermining the efficiency of the solvers when dealing with large size networks. For example, in a network with 1000 edges there are 1 million elements in the covariance matrix associated with every quadratic constraint. Therefore there is a need to develop a solution strategy which is capable of handling large networks efficiently. In this section, an outer approximation algorithm was developed to solve MICQP formulation of one origin to all destination robust shortest path problems for large size tests network. The performance of the proposed solution strategy is not compromised by large covariance matrices, and the algorithm running time is independent of different covariance matrix structures. The solution methodology and more details of the outer approximation algorithm are presented in the next section.

3.3 Outer Approximation Algorithm for the mean-STD shortest path

Lemma1. *The function* $Z^{rs}(x,t) = \sqrt{\sum_{i,j \in E} \sum_{l,k \in E} x_{ij}^{rs} \, cov(a_{ij}, a_{lk}) \, x_{lk}^{rs}} - t^{rs}$ *is convex.*

Proof: Rewrite $Z^{rs} = \sqrt{XQX^T} - t$ using matrix notation, where Q is the covariance matrix. Because Q is positive semidefinite, the Cholesky decomposition $Q = LL^T$ exists and we have:

$$Z = \sqrt{XQX^T} - t = \sqrt{XLL^TX^T} - t = \|XL\| - t \tag{3-9}$$

where $\|.\|$ is the Euclidean norm. Since the second term is linear, it suffices to show that $\|XL\|$ is convex. Consider any vectors X_1 and X_2. We have

$$\begin{aligned} \|(\lambda X_1 + (1-\lambda)X_2)L\| &= \|\lambda X_1 L + (1-\lambda)X_2 L\| \\ &\leq \|\lambda X_1 L\| + \|(1-\lambda)X_2 L\| \\ &= \lambda \, \|X_1 L\| + (1-\lambda) \, \|X_2 L\| \end{aligned}$$

from the triangle inequality, completing the proof. QED

3.3.1 Outer Approximation (OA) subproblem for mean-STD shortest path

As already discussed, the subproblem for OA algorithm is obtained by projecting out the integer variables from the primary MINLP and reducing the formulation to a NLP. More specifically, fixing the binary variable $\hat{x}_{lk}^{rs^h}$ at iteration h in model (P2) would yield the NLP subproblem of OA algorithm for robust shortest path problem. However, in model (P2) given the value for binary variable $\hat{x}_{lk}^{rs^h}$ calculating the optimal value for the continuous variable $\hat{t}^{rs^h}$ is straightforward and can simply achieved by equation (3-10). Consequently, the optimal value for OA subproblem is obtained by equation (3-11). Thus, in the case of robust shortest path problem, it is trivial to solve the NLP OA subproblem to provide an upper bound solution at every iteration using the closed form equation (3-11). Solving the OA subproblem in this way increases computational efficiency, as the NLP subproblem is usually the bottleneck of OA approaches.

$$\hat{t}^{rs^h} = \sqrt{\sum_{a_{ij} \in E} \sum_{a_{lk} \in E} \hat{x}_{ij}^{rs^h} cov(a_{ij}, a_{lk})\, \hat{x}_{lk}^{rs^h}} \qquad \forall s \in S \qquad (3\text{-}10)$$

$$Z^h = Min\; w \sum_{\substack{a_{ij} \in E \\ s \in S}} c_{ij} \hat{x}_{ij}^{rs^h} + (1-w) \sum_{s \in S} \hat{t}^{rs^h} \qquad \forall s \in S \qquad (3\text{-}11)$$

3.3.2 Outer Approximation (OA) master problem

The mixed integer program master problem (MP) however must be solved in order to yield the lower bound solution for OA algorithm. In order to define the MP we need to obtain valid linear approximation of the conic constraints.

Theorem 1

Let $\hat{x}_{ij}^{rs}$ be the integer feasible assignment for OA sub problem and $\hat{t}^{rs}$ be the optimal solution for the nonlinear sub problem of the OA algorithm, equation (3-12) is a valid linear approximation of the conic constraint (3-6).

$$\sum_{a_{ij} \in E} \sigma_{ij}^2 \hat{x}_{ij}^{rs} (x_{ij}^{rs} - \hat{x}_{ij}^{rs}) + \sum_{a_{ij} \in E} \sum_{\substack{a_{lk} \in E \\ l,k \neq i,j}} cov(a_{ij}, a_{lk}) \hat{x}_{lk}^{rs} (x_{ij}^{rs} - \hat{x}_{ij}^{rs}) - \hat{t}^{rs}(t^{rs} - \hat{t}^{rs}) \leq 0 \tag{3-12}$$

Proof:

Assume that point $(\hat{x}_{ij}^{rs}, \hat{t}^{rs})$ is a feasible assignment for problem P2, according to convexity of $Z^{rs}(x,t)$ we can write the linear approximation as:

$$\sqrt{\sum_{a_{ij}\in E} \sigma_{ij}^2 \hat{x}_{ij}^{rs\,2} + \sum_{a_{ij}\in E} \sum_{\substack{a_{lk}\in E \\ l,k\neq i,j}} cov(a_{ij}, a_{lk}) \hat{x}_{ij}^{rs} \hat{x}_{lk}^{rs}}$$

$$+ \frac{\sum_{a_{ij}\in E} 2\sigma_{ij}^2 \hat{x}_{ij}^{rs} (x_{ij}^{rs} - \hat{x}_{ij}^{rs})}{2\sqrt{\sum_{a_{ij}\in E} \sigma_{ij}^2 \hat{x}_{ij}^{rs\,2} + \sum_{a_{ij}\in E} \sum_{\substack{a_{lk}\in E \\ l,k\neq i,j}} cov(a_{ij}, a_{lk}) \hat{x}_{ij}^{rs} \hat{x}_{lk}^{rs}}} \qquad \forall s \in S \qquad (3\text{-}13)$$

$$+ \frac{2\sum_{a_{ij}\in E} \sum_{\substack{a_{lk}\in E \\ l,k\neq i,j}} cov(a_{ij}, a_{lk}) \hat{x}_{lk}^{rs} (x_{ij}^{rs} - \hat{x}_{ij}^{rs^h})}{2\sqrt{\sum_{a_{ij}\in E} \sigma_{ij}^2 \hat{x}_{ij}^{rs\,2} + \sum_{a_{ij}\in E} \sum_{\substack{a_{lk}\in E \\ l,k\neq i,j}} cov(a_{ij}, a_{lk}) \hat{x}_{ij}^{rs} \hat{x}_{lk}^{rs}}} - \hat{t}^{rs}$$

$$- (t^{rs} - \hat{t}^{rs}) \leq 0$$

Considering the fact that $\hat{t}^{rs}$ is the optimal solution to the OA sub problem which is equal to $\hat{t}^{rs} = \sqrt{\sum_{i,j\in E} \sum_{l,k\in E} \hat{x}_{ij}^{rs} cov(a_{ij}, a_{lk}) \hat{x}_{lk}^{rs}}$, equation (3-13) can further be simplified to the following expression as shown below:

$$\hat{t}^{rs} + \frac{\sum_{a_{ij}\in E} 2\sigma_{ij}^2 \hat{x}_{ij}^{rs} (x_{ij}^{rs} - \hat{x}_{ij}^{rs})}{2\,\hat{t}^{rs}} +$$

$$\frac{2\sum_{a_{ij}\in E} \sum_{\substack{a_{lk}\in E \\ l,k\neq i,j}} cov(a_{ij}, a_{lk}) \hat{x}_{lk}^{rs} (x_{ij}^{rs} - \hat{x}_{ij}^{rs})}{2\,\hat{t}^{rs}} - \hat{t}^{rs} - (t^{rs} - \hat{t}^{rs}) \leq 0 \qquad (3\text{-}14)$$

Pre multiplying the above equation by $2\hat{t}^{rs}$ by considering the fact that $\hat{t}^{rs} \neq 0$ the following linear approximation can be achieved.

$$\sum_{a_{ij}\in E} \sigma_{ij}^2 \hat{x}_{ij}^{rs} \left(x_{ij}^{rs} - \hat{x}_{ij}^{rs}\right) + \sum_{a_{ij}\in E} \sum_{\substack{a_{lk}\in E \\ l,k \neq i,j}} cov\left(a_{ij}, a_{lk}\right) \hat{x}_{lk}^{rs} \left(x_{ij}^{rs} - \hat{x}_{ij}^{rs}\right) - \hat{t}^{rs}\left(t^{rs} - \hat{t}^{rs}\right) \leq 0 \tag{3-15}$$

The MP formulation of OA algorithm for robust shortest path problem is provided below:

$$Z = Min\ w \sum_{\substack{a_{ij}\in E \\ s\in S}} c_{ij} x_{ij}^{rs} + (1-w)\sum_{s\in S} t^{rs} \tag{3-16}$$

$$\sum_{j:a_{ij}\in E} x_{ij}^{rs} - \sum_{j:a_{ji}\in E} x_{ji}^{rs} = \begin{cases} 1 & \forall\, i = r \\ 0 & \forall\, i \neq r, i \neq s \\ -1 & \forall\, i = s \end{cases} \qquad \forall s \in S \tag{3-17}$$

$$\sum_{a_{ij}\in E} \sigma_{ij}^2 \hat{x}_{ij}^{rs^h} \left(x_{ij}^{rs} - \hat{x}_{ij}^{rs^h}\right) + \sum_{a_{ij}\in E} \sum_{\substack{a_{lk}\in E \\ l,k \neq i,j}} cov\left(a_{ij}, a_{lk}\right) \hat{x}_{lk}^{rs^h} \left(x_{ij}^{rs} - \hat{x}_{ij}^{rs^h}\right) - \hat{t}^{rs^h}\left(t^{rs} - \hat{t}^{rs^h}\right) \leq 0 \qquad \forall s \in S,\ h = 1,.,H \tag{3-18}$$

$$x_{ij}^{rs} \in \{0,1\}, t^{rs} \geq 0 \qquad \forall a_{ij} \in E,\ s \in S \tag{3-19}$$

The OA master problem is comprised of two main types of constraints: flow balance constraints (24), and the OA inequalities which are basically cutting off those solutions that do not belong to linear approximation of the feasible region at every iteration until the desired convergence is reached. According to constraint (3-18), at every iteration only those elements of the covariance matrix which are constructing the shortest paths are included in the model and there is no need to account for all other elements of the network covariance matrix. This property enables the model to cope with the huge travel time covariance matrix. As mentioned earlier, the lower bound for of the OA algorithm is achieved after solving the mixed integer linear master problem. It should be pointed out that as the algorithm proceeds; non decreasing order of lower bounds will be gained since more OA inequalities (3-18) will be added to the master problem. The steps of the implemented OA algorithm are provided in below.

Table 3-1: Outer Approximation (OA) Framework

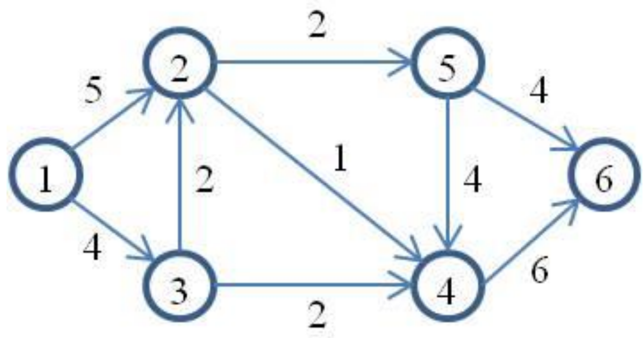
Outer Approximation (OA) Algorithm for one origin to all destination mean-standard deviation shortest path
Input: Convergence tolerance ε, Maximum Number of Iteration *H*,
Upperbound(UB) = $+\infty$ Lowerbound (LB) =-∞, *h=1*
Initialization: Solve One-all Shortest path problem for initial value of $\hat{\boldsymbol{x}}_{lk}^{rs^1}$
1: If (UB-LB) ≤ ε or $\boldsymbol{h} > \boldsymbol{H}$ then go to step 7
2: Calculate $\hat{\boldsymbol{t}}^{rs}$ according to equation (3-10)
3: Calculate the subproblem objective function $\boldsymbol{Z}^{h}$ according to equation (3-11)
4: If ($\boldsymbol{Z}^{h} <$ UP), Update the upperbound and,
update the current best points as $\overline{\boldsymbol{x}}_{ij}^{rs} = \hat{\boldsymbol{x}}_{lk}^{rs^h}$ and $\bar{\boldsymbol{t}}^{rs} = \hat{\boldsymbol{t}}^{rs^h}$
5: Add the OA inequalites and solve the MP based on equation (3-16) to (3-19)
6: Update the LB, *h=h+1*, go to step 1
7:Report $\overline{\boldsymbol{x}}_{ij}^{rs}$ and $\bar{\boldsymbol{t}}^{rs}$, Algorithm Stops

3.4 Numerical Experiment

This section describes computational experiments conducted in order to study the performance of the outer approximation algorithm in solving the one origin to all destinations robust shortest path problem. The tests were conducted on a Dell Optiplex 990 computer with an Intel Pentium i7-2600 CPU at 3.4 GHz processor and 16 GB RAM. CPLEX 12 was selected as the default solver for handling the mixed integer linear programs. Note that the master problem has a network structure with additional constraints. CPLEX 12 has the ability to identify and extract the network constraints and choose appropriate solution algorithms which exploit the network structure. The outer approximation algorithm was fully coded in the GAMS platform.

3.4.1 Illustrative Example

We start the experimental section by showing OA algorithm steps and convergence trajectory for a simple illustrative example comprised of six nodes and nine arcs. Network and the associated covariance matrix are presented in Figure 3-1. Robust shortest path has been calculated for origin (1) through origin (6) for weight parameter w=0.5. The details of algorithm steps including objective value for master and subprobem, lower and upper bound, and the relative tolerance are presented in Table 3-2. The results show that the OA algorithm is completed after finishing two iterations given the initial shortest path solutions. As expected the convergence measurement decreases after the first iteration. Moreover, optimality gap of zero is reached which implies that the global solution is achieved for this example.



	a_{12}	a_{13}	a_{24}	a_{25}	a_{32}	a_{34}	a_{46}	a_{54}	a_{56}
a_{12}	5	1.2	0.75	0.75	-0.325	0.75	0.5	0.7625	1.5
a_{13}	1.2	3	0.875	-0.5	0.55	-0.8	-1.125	0.95	0.7875
a_{24}	0.75	0.875	2	1	1	0.8	-0.3	0.3875	0.5125
a_{25}	0.75	-0.5	1	3	1.25	0.9	0.5	-0.325	1
a_{32}	-0.325	0.55	1	1.25	1	0.7	-0.5	0.375	0.275
a_{34}	0.75	-0.8	0.8	0.9	0.7	3.5	0.625	-0.3625	-0.1375
a_{46}	0.5	-1.125	-0.3	0.5	-0.5	0.625	2	0.8125	1
a_{54}	0.7625	0.95	0.3875	-0.325	0.375	-0.3625	0.8125	3.5	-0.6125
a_{56}	1.5	0.7875	0.5125	1	0.275	-0.1375	1	-0.6125	4

Figure 3-1 Illustrative network with arc cost and the covariance matrix

Table 3-2- OA detail results

<table>
<tr><td colspan="3">Initial Solution</td><td colspan="3">Initialization</td></tr>
<tr><td rowspan="5">O-D Pair</td><td>1-2</td><td>1-2</td><td colspan="3" rowspan="5">UB=+∞
LB=-∞
h=**1**</td></tr>
<tr><td>1-3</td><td>1-3</td></tr>
<tr><td>1-4</td><td>1-2-4</td></tr>
<tr><td>1-5</td><td>1-2-5</td></tr>
<tr><td>1-6</td><td>1-2-5-6</td></tr>
<tr><td colspan="2">Iteration h=1</td><td>Updated Solution</td><td>SP Obj</td><td>UB</td><td>Tol (%)</td></tr>
<tr><td rowspan="5">O-D Pair</td><td>1-2</td><td>1-3-2</td><td rowspan="2">47.920</td><td rowspan="2">47.920</td><td rowspan="5">12.3</td></tr>
<tr><td>1-3</td><td>1-3</td></tr>
<tr><td>1-4</td><td>1-3-4</td><td>MP Obj</td><td>LB</td></tr>
<tr><td>1-5</td><td>1-3-2-5</td><td rowspan="2">42.025</td><td rowspan="2">42.025</td></tr>
<tr><td>1-6</td><td>1-3-4-6</td></tr>
<tr><td colspan="2">Iteration h=2</td><td>Updated Solution</td><td>SP Obj</td><td>UB</td><td>Tol (%)</td></tr>
<tr><td rowspan="5">O-D Pair</td><td>1-2</td><td>1-2</td><td rowspan="2">46.015</td><td rowspan="2">46.015</td><td rowspan="5">0</td></tr>
<tr><td>1-3</td><td>1-3</td></tr>
<tr><td>1-4</td><td>1-3-4</td><td>MP Obj</td><td>LB</td></tr>
<tr><td>1-5</td><td>1-2-5</td><td rowspan="2">46.015</td><td rowspan="2">46.015</td></tr>
<tr><td>1-6</td><td>1-3-4-6</td></tr>
</table>

3.4.2 Test Experiments

In this section four different typical transportation test networks have been selected for investigating the performance of the devised outer-approximation algorithm. The links travel cost for the test networks was obtained from Bar-Gera (2013) according to the best known link flow solutions. Table 3-3 shows the details for network examples.

Table 3-3. Test Networks

Networks	Number of Links(A)	Number of Nodes (N)
Sioux Falls	76	24
Anaheim	914	416
Barcelona	2522	1020
Chicago Sketch	2250	933

In this dissertation we do not impose any spatial limitation on link travel cost correlations; thus any link travel cost can potentially be correlated to the travel cost of every other link throughout the network. The standard deviation of link a_{ij}, σ_{ij} is generated as:

$$\sigma_{ij} = Uniform(0, b)c_{ij} \tag{3-20}$$

where c_{ij} is the travel cost of link a_{ij} and $Uniform(0, b)$ corresponds to a uniform random number between 0 and b. The correlation coefficient is generated as a uniform random number between $-p$ and p. Note that b and p correspond to a real number between 0 and 1. We use the method of Hardin et al. (2012) to generate valid correlation matrices.

The main goal of the experiments is to compare the performance of the proposed outer-approximation algorithm in solving one origin to all destinations robust shortest path problems for different covariance matrix structures. Computation time and relative optimality gap are used as the criteria for this comparison. The algorithm will stop and the best solution will be reported when the desired optimality gap is reached. In these experiments, the relative optimality gap measure is the difference between the objective function of the subproblem and the master problem divided by the objective function of the subproblem. The relative convergence measure of the implemented outer-approximation algorithm was set at 0.1%. Additionally, the algorithm terminates after 100 iterations. The mixed integer master problem of the algorithm was solved within 1% of relative optimality gap. Fletcher and Leyffer (1994) mentioned that it is not essential to solve the master problem to zero optimality gap as long as the MP yields feasible integer solutions. In this dissertation, the mixed integer linear program (MIP) MP is simply a shortest path problem with side constraints, which delivers feasible integer solutions. Different covariance matrices have been tested on the four text networks to indicate the effects of various levels of variance and correlation among the links on the achieved route travel cost. In order to account for the random nature of generated covariance matrices, each test scenario is solved ten times. The maximum, minimum, and average computational time and optimality gap of ten randomly generated solved network instances along with minimum, mode and maximum number of iterations needed to complete the algorithm are reported as the result for each test scenario. Results of the algorithm in terms of number of iterations, running time and the previously defined convergence measurement on the four test examples with weight parameter w equal to 0.5 are presented in Tables (3-4) to (3-7).

The results show that the algorithm running time is not affected by different levels of variance, and the solution algorithm is insensitive to the covariance matrix structure. Furthermore, the difference between the minimum, average and maximum amount of running time of ten different randomly generated instances at every test scenario is also negligible. As expected, the algorithm running time starts to grow with the network size, from less than a second for Sioux Falls to nearly two minutes for the Barcelona network. Convergence criteria of zero are reported in many instances which verify the ability of the algorithm in returning the optimal solution. In the case of Sioux Falls network the algorithm is able to deliver optimal solution in less than a second; optimal solutions are achieved for Anaheim network in approximately 6 seconds. For the cases of Barcelona and Chicago Sketch optimality gap of very close to zero is reported. Furthermore, in majority of the tests scenarios the desired level of optimality gap is achieved after completing two iterations of OA algorithm. However, for networks like Chicago Sketch and Barcelona which are the large test networks three iterations are required to achieved the solutions in the case of high variance and correlation factors. Thus, the results confirm the fact that OA algorithm requires few iterations in order to yield the optimal solution.

Table 3-4. Sioux Falls network results

Variance Factor	Correlation Factor	Number of Iterations			Running Time (Sec)			Relative Convergence (%)		
b	p	Min	Mode	Max	Min	Ave	Max	Min	Ave	Max
0.1	0	2	2	2	0.12	0.15	0.23	0	0	0
	0.1	2	2	2	0.12	0.19	0.24	0	0	0
	0.2	2	2	2	0.12	0.15	0.23	0	0	0
	0.3	2	2	2	0.12	0.17	0.24	0	0	0
	0.4	2	2	2	0.12	0.15	0.24	0	0	0
	0.5	2	2	2	0.12	0.18	0.24	0	0	0
	0.6	2	2	2	0.13	0.15	0.23	0	0	0
	0.7	2	2	2	0.12	0.16	0.23	0	0	0
	0.8	2	2	2	0.12	0.14	0.23	0	0	0
	0.9	2	2	2	0.12	0.17	0.24	0	0	0
0.3	0	2	2	2	0.13	0.17	0.24	0	0	0
	0.1	2	2	2	0.13	0.16	0.23	0	0	0
	0.2	2	2	2	0.12	0.16	0.44	0	0	0
	0.3	2	2	2	0.12	0.14	0.23	0	0	0
	0.4	2	2	2	0.13	0.17	0.24	0	0	0
	0.5	2	2	2	0.12	0.16	0.23	0	0	0
	0.6	2	2	2	0.12	0.15	0.23	0	0	0
	0.7	2	2	2	0.12	0.14	0.23	0	0	0
	0.8	2	2	2	0.12	0.15	0.23	0	0	0
	0.9	2	2	2	0.13	0.15	0.23	0	0	0
0.5	0	2	2	2	0.12	0.15	0.23	0	0	0
	0.1	2	2	2	0.13	0.15	0.22	0	0	0
	0.2	2	2	2	0.12	0.16	0.24	0	0	0
	0.3	2	2	2	0.12	0.15	0.23	0	0	0
	0.4	2	2	2	0.12	0.15	0.23	0	0	0
	0.5	2	2	2	0.12	0.15	0.24	0	0	0
	0.6	2	2	2	0.12	0.15	0.23	0	0	0
	0.7	2	2	2	0.12	0.19	0.24	0	0	0
	0.8	2	2	2	0.12	0.15	0.23	0	0	0
	0.9	2	2	2	0.12	0.17	0.24	0	0	0

Table 3-5 Anaheim network results

Variance Factor	Correlation Factor	Number of Iterations			Running Time (Sec)			Relative Convergence(%)		
b	p	Min	Mode	Max	Min	Ave	Max	Min	Ave	Max
0.1	0	2	2	2	5.62	5.64	5.68	0	0	0
	0.1	2	2	2	5.60	5.62	5.63	0	0	0
	0.2	2	2	2	5.60	5.62	5.63	0	0	0
	0.3	2	2	2	5.60	5.63	5.66	0	0	0
	0.4	2	2	2	5.60	5.63	5.66	0	0	0
	0.5	2	2	2	5.60	5.62	5.65	0	0	0
	0.6	2	2	2	5.60	5.62	5.63	0	0	0
	0.7	2	2	2	5.60	5.62	5.66	0	0	0
	0.8	2	2	2	5.60	5.62	5.63	0	0	0
	0.9	2	2	2	5.62	5.65	5.69	0	0	0
0.3	0	2	2	2	5.63	5.66	5.69	0	0	0
	0.1	2	2	2	5.63	5.67	5.69	0	0	0
	0.2	2	2	2	5.62	5.66	5.69	0	0	0
	0.3	2	2	2	5.66	5.68	5.71	0	0	0
	0.4	2	2	2	5.65	5.67	5.68	0	0	0
	0.5	2	2	2	5.66	5.68	5.69	0	0	0
	0.6	2	2	2	5.65	5.67	5.69	0	0	0
	0.7	2	2	2	5.65	5.68	5.69	0	0	0
	0.8	2	2	2	5.69	5.71	5.73	0	0	0
	0.9	2	2	2	5.69	5.76	5.87	0	0	0
0.5	0	2	2	2	5.71	5.77	5.88	0	0	0
	0.1	2	2	2	5.68	5.70	5.71	0	0	0
	0.2	2	2	2	5.71	5.72	5.74	0	0	0
	0.3	2	2	2	5.69	5.72	5.74	0	0	0
	0.4	2	2	2	5.68	5.73	5.80	0	0	0
	0.5	2	2	2	5.68	5.71	5.73	0	0	0
	0.6	2	2	2	5.69	5.73	5.76	0	0	0
	0.7	2	2	2	5.62	5.64	5.68	0	0	0
	0.8	2	2	2	5.60	5.62	5.63	0	0	0
	0.9	2	2	2	5.60	5.62	5.63	0	0	0

Table 3-6. Chicago sketch Results

Variance Factor	Correlation Factor	Number of Iterations			Running Time (Sec)			Relative Convergence (%)		
b	p	Min	Mode	Max	Min	Ave	Max	Min	Ave	Max
0.1	0	2	2	2	56.10	58.74	59.44	0.02	0.04	0.09
	0.1	2	2	2	56.80	58.83	59.29	0.02	0.06	0.07
	0.2	2	2	2	56.29	58.74	59.64	0.03	0.04	0.10
	0.3	2	2	2	56.65	58.80	59.94	0.03	0.07	0.08
	0.4	2	2	2	56.38	58.63	59.88	0.02	0.06	0.08
	0.5	2	2	2	56.73	58.55	59.01	0.03	0.07	0.08
	0.6	2	2	2	56.32	58.65	59.83	0.02	0.06	0.10
	0.7	2	2	2	56.85	58.60	59.37	0.02	0.03	0.10
	0.8	2	2	2	56.55	58.97	59.83	0.03	0.04	0.10
	0.9	2	2	2	57.47	59.10	60.63	0.03	0.05	0.08
0.3	0	2	2	2	57.84	59.29	60.01	0.01	0.03	0.08
	0.1	2	2	2	57.80	59.66	60.11	0.02	0.07	0.09
	0.2	2	2	2	57.89	59.52	60.01	0.02	0.07	0.08
	0.3	2	2	2	58.00	59.60	60.83	0.03	0.05	0.08
	0.4	2	2	2	57.48	59.49	60.94	0.01	0.05	0.10
	0.5	2	2	2	57.20	59.72	60.40	0.02	0.04	0.09
	0.6	2	2	2	57.87	59.68	60.74	0.02	0.06	0.09
	0.7	2	2	2	58.83	60.46	60.53	0.02	0.04	0.10
	0.8	2	2	3	58.76	60.38	61.65	0.01	0.04	0.07
	0.9	2	2	3	58.83	60.19	61.39	0.02	0.07	0.09
0.5	0	2	2	2	58.85	60.11	61.11	0.02	0.05	0.09
	0.1	2	2	2	58.12	60.36	61.95	0.02	0.05	0.09
	0.2	2	2	2	58.64	60.46	61.81	0.02	0.03	0.10
	0.3	2	2	2	58.09	60.61	61.65	0.01	0.03	0.08
	0.4	2	2	2	58.02	60.53	61.16	0.03	0.05	0.08
	0.5	2	2	2	58.30	60.58	61.12	0.01	0.05	0.08
	0.6	2	2	2	58.74	60.54	61.00	0.02	0.05	0.09
	0.7	2	2	2	58.19	60.34	61.10	0.02	0.05	0.09
	0.8	2	2	3	58.14	60.56	61.40	0.03	0.03	0.09
	0.9	2	2	3	58.14	60.31	61.20	0.02	0.04	0.07

Table 3-7. Barcelona Network Results

Variance Factor	Correlation Factor	Number of Iterations			Running Time (Sec)			Relative Convergence (%)		
b	P	Min	Mode	Max	Min	Ave	Max	Min	Ave	Max
0.1	0	2	2	2	154.79	155.54	158.03	0.01	0.07	0.10
	0.1	2	2	2	154.54	155.97	158.2	0.02	0.06	0.10
	0.2	2	2	2	154.89	155.83	158.19	0.03	0.07	0.09
	0.3	2	2	2	154.41	155.99	158.25	0.03	0.07	0.10
	0.4	2	2	2	154.93	156.26	159.1	0.01	0.08	0.09
	0.5	2	2	2	154.56	155.65	159.01	0.04	0.07	0.10
	0.6	2	2	2	154.17	156.29	159.46	0.05	0.07	0.09
	0.7	2	2	2	154.68	155.35	159.11	0.04	0.08	0.10
	0.8	2	2	2	154.51	155.51	159.27	0.03	0.06	0.08
	0.9	2	2	3	154.72	157.54	159.13	0.01	0.08	0.09
0.3	0	2	2	2	154.2	157.36	159.16	0.04	0.08	0.10
	0.1	2	2	2	155.13	156.85	159.45	0.02	0.06	0.09
	0.2	2	2	2	155.03	157.71	159.15	0.03	0.07	0.08
	0.3	2	2	2	155.63	157.74	159.29	0.03	0.05	0.09
	0.4	2	2	2	156.46	157.82	159.25	0.05	0.06	0.09
	0.5	2	2	2	155.37	158.63	159.03	0.05	0.08	0.10
	0.6	2	2	2	156.37	158.22	160.42	0.02	0.07	0.08
	0.7	2	2	2	155.56	157.72	160.75	0.04	0.06	0.10
	0.8	2	2	3	155.76	158.74	160.79	0.02	0.08	0.09
	0.9	2	2	3	155.72	158.83	160.98	0.03	0.08	0.10
0.5	0	2	2	2	155.09	160.19	158.74	0.03	0.07	0.10
	0.1	2	2	2	155.51	160.64	158.83	0.02	0.06	0.08
	0.2	2	2	2	155.37	161.95	160.11	0.04	0.06	0.09
	0.3	2	2	2	155.63	160.11	161.82	0.02	0.07	0.08
	0.4	2	2	2	156.7	159.27	161.47	0.02	0.07	0.08
	0.5	2	2	2	155.62	160.11	161.92	0.01	0.07	0.08
	0.6	2	2	3	155.31	160.17	161.61	0.01	0.07	0.08
	0.7	2	2	3	156.4	160.31	161.62	0.03	0.08	0.09
	0.8	2	2	3	156.95	160.59	161.2	0.05	0.07	0.09
	0.9	2	3	3	155.93	160.53	161.76	0.04	0.06	0.10

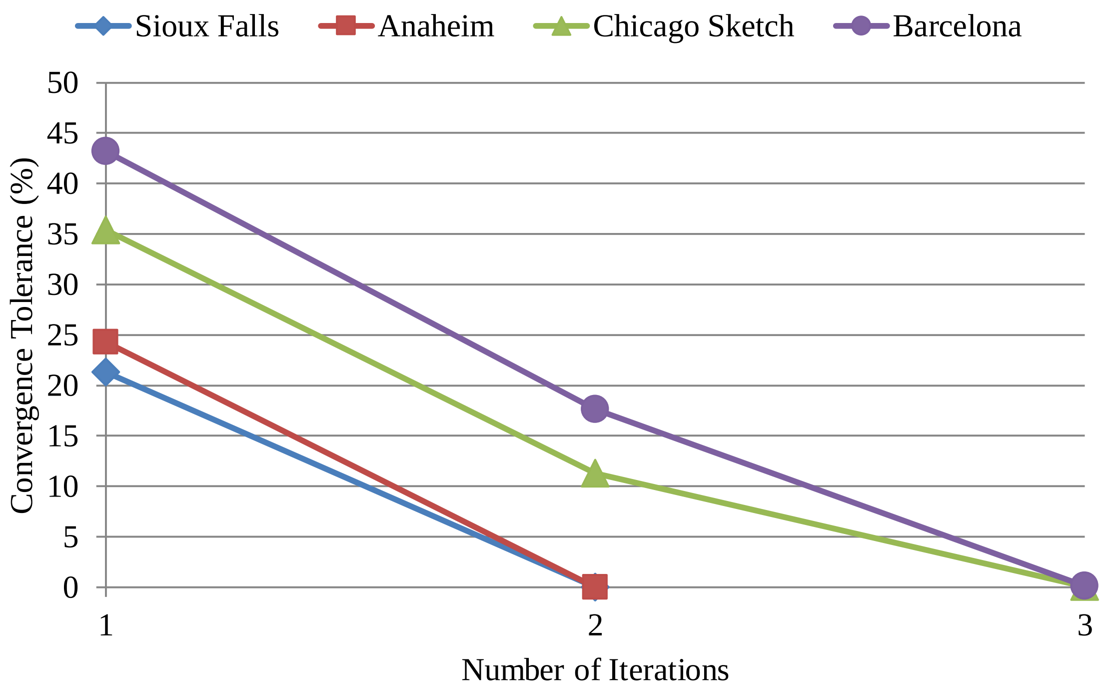


Figure 3-2-OA performance based on number of iterations

Convergence trajectory of the OA algorithm has been depicted in Figure 3-2 for four test networks considered in this dissertation. We assume that the standard deviation $\sigma_{ij}$for link a_{ij}, is generated as $\sigma_{ij} = Uniform(0,0.5)c_{ij}$ and the correlation coefficient is generated as a random number between -0.9 and 0.9. Robust shortest path was solved for 10 randomly generated covariance matrices and the average of convergence tolerance for every iteration is plotted. As we can see from the results OA algorithm would converge to the optimal solutions in only two iterations for Sioux Falls and Anaheim networks. Furthermore, in this test experiment the optimality gap of less than 0.1% was reached in three iterations for the two networks of Chicago Sketch and Barcelona.

The purpose of the next set of experiments is to compare the efficiency of the outer approximation algorithm with current available branch and cut based solver packages for solving MICQP types of problems. In this test experiments, we just considered robust shortest path with uncorrelated link travel cost due to the significantly high number of nonlinear terms associated with every quadratic constraint in direct implementation of the model (P2) as MICQP in GAMS, This is mostly because CPLEX solver was unable to handle large size link travel cost covariance matrices of networks like Chicago Sketch and Barcelona Therefore we assume that the standard deviation $\sigma_{ij}$for link a_{ij}, is generated as $\sigma_{ij} = Uniform(0,0.5)c_{ij}$ and the correlation coefficient is considered to be zero. Convergence criteria were set at 0.1% for both the outer approximation and MICQP formulation. Similar to the outer approximation algorithm, the MICQP formulation for robust shortest path problem was implemented in GAMS and then solved by CPLEX 12. The average results of ten randomly generated instances of the two approaches on the four test example show that the outer approximation algorithm clearly outperforms the MICQP formulation. While the difference in running time between the two approaches is negligible for small networks, this difference starts to grow drastically with increase in network size. For instance, MICQP formulation for the Barcelona networks delivers the solution in 1235.6 seconds whereas the same solution is achieved in almost 120 seconds by outer approximation algorithm. Therefore, even in the case of no correlation assumption on link travel cost which is considered as an easier case when compared to the correlated case, OA approach leads to significantly faster solutions when compared to solutions obtained by solving the MICQP directly through the GAMS/CPLEX package. Figure 3 illustrates the running time for both approaches on the four test networks.

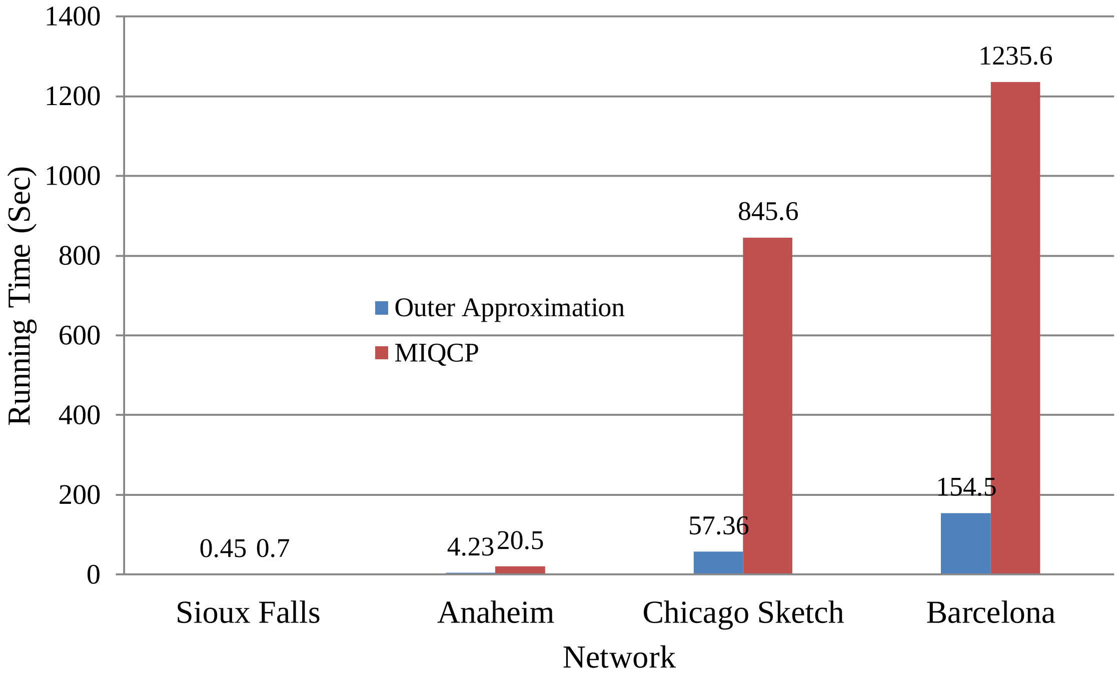
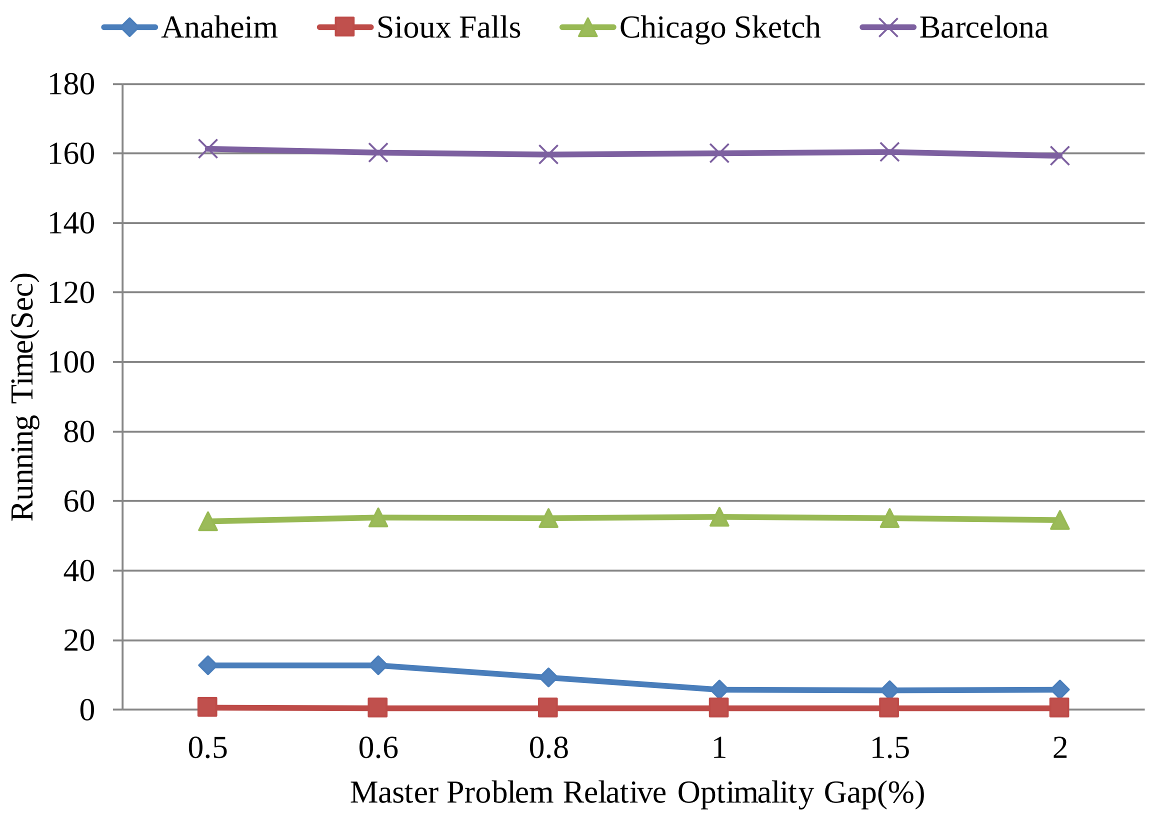


Figure 3-3- MICQP formulation versus Outer Approximation algorithm

An additional set of experiments was carried out to study the effects of master problem precision on algorithm running time. Six different relative optimality gaps were selected for solving the master problem in order to achieve to this purpose. In this experiment, we assume that the standard deviation σ_{ij} for link a_{ij} , is generated as $\sigma_{ij} = Uniform(0{,}0.5)c_{ij}$ and the correlation coefficient is generated as a random number between -0.9 and 0.9. Likewise the previous experiments, the test were coded and solved by GAMS/CPLEX 12. The results presented in Figure



4 show that the running time for Sioux Falls, Chicago Sketch and Barcelona networks were remained unchanged for different master problem precision levels, while slight changes in running time were observed for Anaheim network for optimality gaps of less than 1%. Thus, these results prove that choosing 1% as the relative optimality gap for the outer approximation master problem is a reasonable assumption in this dissertation, and different precision would not significantly alter the running time of the outer approximation algorithm.

Figure 3-4- Performance of the algorithm with different relative optimality gap for master problem

The sensitivity analysis of weight parameter w is performed in the final set of experiments. In this part, the shortest path problem is solved for all test networks based on five different values of the weight parameter w. The correlation matrix is generated similarly to the previous experiment with $\sigma_{ij} = Uniform(0,0.5)c_{ij}$ and the correlation coefficient is generated as a random number between -0.9 and 0.9. The results are presented as the average computational time of ten randomly generated covariance matrix associated with every level of weight parameter w. Figure 3-5 shows that computation time tends to increase with the weight placed on the cost standard deviation. This is mainly because placing heavy weight on standard deviation term which is the complex part of the formulation would require extra computational cost in order to achieve to the desired level of optimality. This increase as expected is more prominent for larger networks like Barcelona compared to smaller networks like Sioux Falls.

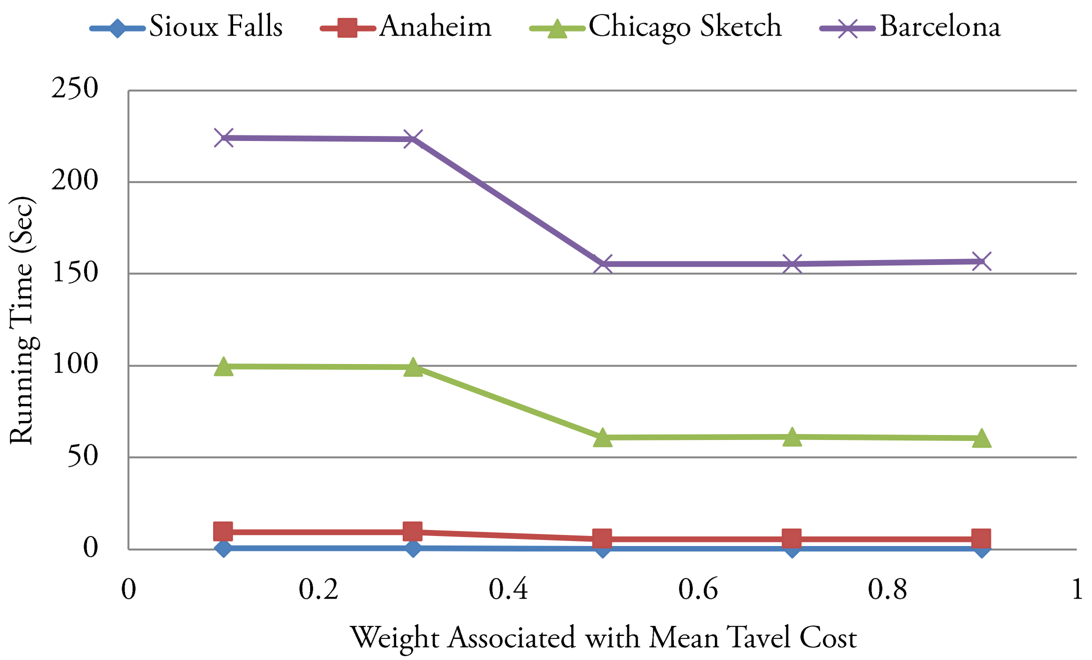


Figure 3-5 Performance of the algorithm with different levels of weight parameter *w*

In addition, the Outer Approximation Algorithm was implemented for the mean variance shortest path problem and compared with the algorithm developed by Sen et al. (2001). The approach proposed by Sen et al (2001) is a three-step method for solving the mean variance shortest path problem, the first of which involves solution of a relaxed quadratic program. From our results, even this first step requires substantially more time than complete solution with our proposed OA algorithm. Sioux Falls and Anaheim were selected as networks for testing procedure. We considered the general covariance matrix with randomly generated standard deviation and correlation coefficients. Tables 3-8 and 3-9 display the average computational time for 10 randomly generated instances solved by OA and relaxed quadratic program for mean variance shortest path problem.

Table 3-8. Sioux Falls Computational time for one origin to all destinations. p=0.1

	OA-Algorithm	Relaxed Quadratic Program
Variance Factor	Time (sec)	Time (sec)
b=0.1	0.047	0.201
b=0.2	0.046	0.205
b=0.3	0.047	0.203
b=0.4	0.032	0.218
b=0.5	0.046	0.225
b=0.6	0.048	0.234
b=0.7	0.050	0.230
b=0.8	0.061	0.233
b=0.9	0.053	0.230

Table 3-9. Sioux Falls Computational time for one origin to all destinations. p=0.5

	OA-Algorithm	Relaxed Quadratic Program
Variance Factor	Time (sec)	Time (sec)
b=0.1	0.048	0.203
b=0.2	0.046	0.203
b=0.3	0.047	0.208
b=0.4	0.045	0.210
b=0.5	0.044	0.218
b=0.6	0.048	0.215
b=0.7	0.051	0.241
b=0.8	0.061	0.239
b=0.9	0.061	0.245

The experiment was then repeated for the Anaheim network; however, CPLEX was unable to generate and solve the relaxed quadratic program in a reasonable amount of time, largely due to the great number of nonlinear terms associated with the quadratic part of the objective function. For example, with the Anaheim network (which has 917) links there are 840,889 elements associated with every origin destination pair and accordingly, for one origin to all destinations mean variance shortest path, the number of nonlinear terms will increase to 771,095,213 which is prohibitive.

However, in order to be able to make comparison with the OA algorithm we considered two more other scenarios and tested the two approaches based on the new settings. In the first test experiments we only considered the mean variance shortest path for one origin and a hundred destinations and solved the shortest path with the full correlation matrix. In the second scenario we assumed only limited correlation among the network links. More specifically we assumed that the travel cost for every link is only correlated to the travel cost of adjacent links emanating from the downstream link node. Similar to the test for Sioux Falls network we solved the model for ten randomly generated instances and report the average computational time for two methods in Tables 3-10 to 3-13. The results achieved from running the two models showed that the OA algorithm clearly outperforms solving the relaxed quadratic program which was proposed by Sen et al. (2001). This difference in computational time between the two methods seems to be more significant when considering the complete covariance matrix on the network.

Table 3-10 - Computational time for Anaheim network. 100-O-D pairs, p=0.1

	OA-Algorithm	Relaxed Quadratic Program
Variance Factor	Time (sec)	Time (sec)
b=0.1	3.368	620.728
b=0.2	3.648	741.758
b=0.3	3.273	726.481
b=0.4	2.050	634.398
b=0.5	3.226	644.892
b=0.6	2.279	705.951
b=0.7	3.851	697.324
b=0.8	3.809	683.794
b=0.9	2.200	736.863

Table 3-11- Computational time for Anaheim network. 100-O-D pairs, p=0.5

	OA-Algorithm	Relaxed Quadratic Program
Variance Factor	Time (sec)	Time (sec)
b=0.1	2.357	655.220
b=0.2	3.638	610.050
b=0.3	2.357	621.680
b=0.4	2.124	626.937
b=0.5	3.685	715.545
b=0.6	3.451	622.413
b=0.7	3.524	680.195
b=0.8	2.684	743.688
b=0.9	3.726	776.994

Table 3-12 Computational time for Anaheim network with limited covariance matrix structure,p=0.1

	OA-Algorithm	Relaxed Quadratic Program
Variance Factor	Time (sec)	Time (sec)
b=0.1	5.397	41.979
b=0.2	5.377	41.207
b=0.3	5.423	42.588
b=0.4	5.553	41.251
b=0.5	5.428	41.012
b=0.6	5.579	40.997
b=0.7	5.517	42.854
b=0.8	5.554	41.145
b=0.9	5.559	41.254

Table 3-13 Computational time for Anaheim network with limited covariance matrix structure, p=0.5

	OA-Algorithm	Relaxed Quadratic Program
Variance Factor	Time (sec)	Time (sec)
b=0.1	5.351	40.092
b=0.2	5.476	41.823
b=0.3	5.351	40.826
b=0.4	5.319	42.104
b=0.5	5.648	41.331
b=0.6	5.554	41.873
b=0.7	5.491	42.320
b=0.8	5.523	40.342
b=0.9	5.492	41.434

Thus the OA algorithm was found to outperform the first step of the Sen et al. (2001)'s method. We did not implement step 2 and 3 of Sen et al. (2001)'s method as OA outperformed step 1 itself. We believe the primary reason for this performance involves the correlation matrix. At every iteration of the OA algorithm, only those elements of the covariance matrix corresponding to the shortest paths are included in the model and there is no need to account for all other elements of the network covariance matrix.

Chapter 4 Robust Optimization Strategy for the Shortest Path Problem under Uncertain Link Travel Cost Distribution

4.1 Introduction

As discussed in more detail in the literature review, the assumption of deterministic costs can be relaxed in a multitude of ways. Particularly, the classical formulation is no longer well-defined, because the cost of a path is stochastic. The most direct generalization is to find the least expected-cost path, but in many contexts minimizing risk or cost variability is equally important. Researchers have developed a number of methods to account for these, but such methods generally require exact specifications of the link cost probability distributions, data which can be difficult to obtain in practice.

The field of robust optimization has largely arisen in response to these concerns. By adopting a "minimax" perspective in which the decision maker optimizes against a suitably defined "worst case" scenario, the solution can account for risk aversion, while only requiring the specification of bounds on stochastic parameters, not exact distributions. In particular, given the assumption of uncertainty in cost we consider the link travel cost distribution function to be unknown and, therefore, cannot be fully specified. Structurally, we define the link travel cost as an affine combination of a known mean and several independent variables belong to an ellipsoidal uncertainty set [Chen et al. 2007]. The robust optimization formulation of the shortest path problem is the topic of this chapter.

More specifically, this chapter's contributions are as follows: First, a novel formulation of the robust shortest path problem, which allows for the scenarios to be tied directly to the causes of uncertainty in transportation (such as incidents or poor weather), and captures potential cost correlations among links in a logical manner is presented. The aim of the proposed robust optimization framework is to protect the shortest path decisions against the worst case realization of links cost considering ellipsoidal uncertainty set. Second, the proposed binary nonlinear integer program (BNIP) for the robust model is reformulated as a mixed integer conic quadratic program (MICQP) and outer-approximation algorithm which can efficiently solve this formulation is suggested to tackle this program. Since the formulation presented here is a generalization of those typically used for the robust shortest path problem, this outer-approximation method can be fruitfully applied to these formulations as well.

4.2 Problem Statement

Consider a graph *G(V,E)* where *V* denotes the set of nodes and *E* the set of directed links. Let a_{ij} represent the link connecting node i to node *j*. Assume that nodes $r \subseteq V$ and $s \subseteq V$ denote the origin and destination, respectively. The aim of this chapter is to establish a robust modeling formulation for the shortest path problem where the link travel cost $\tilde{c}_{ij}$ is assumed to be uncertain with unknown distribution. This is a common situation for decision makers, as it is often difficult to accurately estimate even the first and second moments of the travel costs, let alone the full density function. Therefore, from a modeling standpoint there is a need to determine the shortest path where only partial information of the link travel costs distribution parameters is available.

In this research we assume that the link travel cost for every link a_{ij} is described by mean $\bar{c}_{ij}$ and an associated uncertainty term. Mathematically the uncertainty term is defined as the combination of a number of independent variables with an associated weight in which each variable represents one source of uncertainty affecting link travel costs. Hence, the uncertain travel cost $\tilde{c}_{ij}$ is expressed as below [Chen et al. 2007]:

$$\tilde{c}_{ij} = \bar{c}_{ij} + \sum_{k=1}^{K} b_{ijk} \tilde{y}_k \qquad \forall a_{ij} \in E \tag{4-1}$$

where b_{ijk} is the weights associated with K random variables $\tilde{y}_k$, each representing one source of uncertainty. In addition, $\tilde{y}_k$ is considered to be an independent random variable satisfying the following two underlying assumptions:

i) $E(\tilde{y}_k) = 0$ and $Var(\tilde{y}_k^2) = 1$
ii) $\tilde{y}_k$ are all independent.

In addition we make the assumption that the uncertain link costs are always positive. In this work we assume that there are K sources of uncertainty for travel cost in each link. Each source of uncertainty, such as poor weather, incidents at different locations, or fuel prices, is reflected with one random variable y_k. Representing uncertainty in this way allows the impacts of each of these phenomena on the network to be captured in a distinct way through the b_{ijk} terms (e.g., fuel prices may affect network costs on every link, whereas an incident may only affect costs on a small number of nearby links), providing a natural way to reflect link cost correlations and facilitating a "scenario-based" interpretation of the findings. Defining the uncertain travel cost according to expression (4-1) will allow for implicitly modeling correlation between link travel costs, as the covariance between link travel costs can be derived based on expression (4-2) given the assumption (i) to (ii) for random variable $\tilde{y}_k$.

$$Cov(c_{ij}, c_{lm}) = \sum_{k=1}^{K} b_{ijk} b_{lmk} \qquad \forall a_{ij}, a_{lm} \in E \qquad (4\text{-}2)$$

Note that estimation of the correlations between travel times is beyond the scope of this work [Ramezani and Geroliminis, 2012]. In addition we assume that the uncertain random variables are bounded by an ellipsoidal uncertainty set $U = \{\tilde{y}_k, \|y\| \leq \Omega\}$ as described by Bertsimas and Sims [2004]. The budget of uncertainty is an input parameter which will reflect the level of conservatism required. The robust shortest path problem is to find the shortest path under the worst case outcomes of the uncertain variables in the uncertainty set. In the above set Ω corresponds to the budget of uncertainty, which adjusts the desired level of robustness and controls the degree of conservatism of the solution. The size of the uncertainty set is a reflection of the uncertainty protection needed for the shortest path. For example the choice of $\Omega = 0$ would result in the classical shortest path problem. This is because when the budget of uncertainty is zero, then no variation in travel cost is allowed. The classical shortest path is solved for the mean travel cost and does not account for any changes in travel cost due to uncertainty. As the budget of uncertainty increases, larger variations in travel cost from mean travel cost are allowed. If a larger uncertainty budget is considered, then the corresponding shortest path will be more protected against worst case outcomes of the travel cost and the solution would be more conservative. More particularly, higher margins of uncertainty budget would imply that the solution is more stable regarding the realized uncertainties and thus is robust. Considering the uncertain travel cost $\tilde{c}_{ij}$ and ellipsoidal uncertainty set U for variable $\tilde{y}_k$, the robust modeling formulation for the shortest path problem can be derived as below:

$$Z = \min_{X} \max_{\|y\| \leq \Omega} \sum_{(i,j) \in E} \sum_{(r,s)} (\bar{c}_{ij} + \sum_{k=1}^{K} b_{ijk} y_k) x_{ij}^{rs} \tag{4-3}$$

s.t.:

$$\sum_{j:a_{ij} \in E} x_{ij}^{rs} - \sum_{j:a_{ji} \in E} x_{ji}^{rs} = \begin{cases} 1 \; \forall i = r \\ 0 \; \forall i \neq r, i \neq s \\ -1 \forall i = s \end{cases} \qquad \forall r, s \in V \tag{4-4}$$

$$x_{ij}^{rs} \in \{0,1\} \qquad \forall a_{ij} \in E, r \in V, s \in V \tag{4-5}$$

Note that in the above formulation, x_{ij}^{rs} is a binary decision variable equal to one if and only if link a_{ij} lies on the path for origin destination (O-D) pair (*r,s*) and $X = \{x_{ij}^{rs}; \forall a_{ij} \in E, r \in V, s \in V\}$. Equation (4-4) represents the flow balance constraints for a path between origin node *r* and destination node *s*. Given the assumption of positive link costs, solutions with cycles will be suboptimal. This formulation is a minimax nonlinear program which is difficult to handle. However, the ellipsoidal uncertainty set can be transformed to a more structurally attractive types of program, such as conic quadratic programs (CQP's) [Chen et al. 2007]. In order to transform the formulation to a CQP we first need to solve the inner maximization program as written in equation (4-6).

$$Z = \min_{X} \sum_{(i,j) \in E} \sum_{(r,s)} \left(\bar{c}_{ij} x_{ij}^{rs} + \max_{\|y\| \leq \Omega} \sum_{k=1}^{K} b_{ijk} y_k \, x_{ij}^{rs} \right) \tag{4-6}$$

The inner maximization problem involves maximizing Z_2 with respect to y_k over the uncertainty set $\|y\| \leq \Omega$ which is written as equation (4-7).

$$Z_2 = \max_{\|y\| \leq \Omega} \sum_{(i,j) \in E} \sum_{(r,s)} \sum_{k=1}^{K} b_{ijk} y_k \, x_{ij}^{rs} \tag{4-7}$$

There are a number of methods for solving the above maximization problem. We adopted a Lagrangian relaxation strategy in order to provide a closed form solution for the above formulation [Gulpinar et al. 2013]. Assuming the Lagrangian multiplier $\alpha \geq 0$ associated with the constraint $\|y\| \leq \Omega$, the Lagrangian function is

$$L(y, \alpha) = \min_{y_k} \max_{\alpha} \sum_{(i,j) \in E} \sum_{(r,s)} \sum_{k=1}^{K} -b_{ijk} y_k \, x_{ij}^{rs} \; + \alpha(\|y\| - \Omega) \tag{4-8}$$

The optimal value for the Lagrangian function (4-8) can be determined by the first order optimality condition

$$\frac{\partial L(y, \alpha)}{\partial y_k} = \sum_{(i,j) \in E} \sum_{(r,s)} b_{ijk} x_{ij}^{rs} \; - \alpha \frac{y_k}{\|y\|} = 0 \tag{4-9}$$

Therefore, the optimal value for y_k can be calculated as:

$$y_k = \frac{\|y\|}{\alpha} \sum_{(i,j) \in E} \sum_{(r,s)} b_{ijk} x_{ij}^{rs} \qquad\qquad k \in \; K \tag{4-10}$$

Considering the complementary condition $\alpha(\|y\| - \Omega) = 0$ and assuming $\alpha \neq 0$, we can set $\|y\| = \Omega$. Therefore,

$$y_k = \frac{\Omega}{\alpha} \sum_{(i,j)\in E} \sum_{(r,s)} b_{ijk} x_{ij}^{rs} \qquad k \in K \tag{4-11}$$

Given the expression for y_k , $\|y\|$ can be derived as:

$$\|y\| = \frac{\Omega}{\alpha} \sqrt{\sum_k \left(\sum_{(i,j)\in E} \sum_{(r,s)} b_{ijk} x_{ij}^{rs} \right)^2} \tag{4-12}$$

Consequently, the optimal value for Lagrangian multiplier α is:

$$\alpha = \sqrt{\sum_k \left(\sum_{(i,j)\in E} \sum_{(r,s)} b_{ijk} x_{ij}^{rs} \right)^2} \tag{4-13}$$

Substituting α into the equation (4-11) yields the optimal value for y_k as presented below:

$$y_k = \Omega \frac{\sum_{(i,j)\in E} \sum_{(r,s)} b_{ijk} x_{ij}^{rs}}{\sqrt{\sum_k \left(\sum_{(i,j)\in E} \sum_{(r,s)} b_{ijk} x_{ij}^{rs} \right)^2}} \qquad \forall k \in K \tag{4-14}$$

Finally, by inserting the optimal value for y_k in equation (4-7) the optimal objective function value for the inner maximization problem is:

$$Z_2 = \Omega \frac{\sum_k \left(\sum_{(i,j)\in E} \sum_{(r,s)} b_{ijk} x_{ij}^{rs} \right)^2}{\sqrt{\sum_k \left(\sum_{(i,j)\in E} \sum_{(r,s)} b_{ijk} x_{ij}^{rs} \right)^2}} = \Omega \sqrt{\sum_k \left(\sum_{(i,j)\in E} \sum_{(r,s)} b_{ijk} x_{ij}^{rs} \right)^2} \tag{4-15}$$

Substituting the closed form solution for the optimal value of inner maximization problem (4-7) into equation (4-6) the minimax objective function (4-3) is reformulated as a minimization problem (P1):

P1:

$$Z = Min \sum_{(i,j)\in E} \sum_{(r,s)} \bar{c}_{ij} x_{ij}^{rs} + \Omega \sqrt{\sum_{k} \left(\sum_{(i,j)\in E} \sum_{(r,s)} b_{ijk} x_{ij}^{rs} \right)^2} \qquad (4\text{-}16)$$

s.t.:

$$\sum_{j:a_{ij}\in E} x_{ij}^{rs} - \sum_{j:a_{ji}\in E} x_{ji}^{rs} = \begin{cases} 1 \ \forall i = r \\ 0 \ \forall i \neq r, i \neq s \\ -1 \forall i = s \end{cases} \qquad \forall r, s \in V \qquad (4\text{-}17)$$

$$x_{ij}^{rs} \in \{0,1\} \quad \forall a_{ij} \in E, r \in V, s \in V \qquad (4\text{-}18)$$

The above model (P1) is a binary integer nonlinear programming (BINLP) formulation for the robust shortest path problem. However, in order to better utilize the structure of formulation we further transform model P1 to a mixed integer conic quadratic program (MICQP) model P2 by introducing the auxiliary variable t_k and h. We should point out that in the new formulation P2 t_k is a free variable while h is defined as a positive variable. Therefore by introducing the constraint $\sum_{(i,j)\in E} \sum_{(r,s)} b_{ijk} x_{ij}^{rs} \leq t_k$, the nonlinear term in the objective function in model (P1) can be formulated by constraints (4-20) and (4-21) in a new model (P2). A similar strategy was adopted by Atamturk et al. [2012] in order to provide the conic programming counterpart of joint inventory and facility location problems.

P2:

$$Z = \text{Min} \sum_{(i,j)\in E} \sum_{(r,s)} \bar{c}_{ij} x_{ij}^{rs} + \Omega h \tag{4-19}$$

s.t.:

$$\sum_{(i,j)\in E} \sum_{(r,s)} b_{ijk} x_{ij}^{rs} \leq t_k \qquad k \in K \tag{4-20}$$

$$\sqrt{\sum_k {t_k}^2} \leq h \tag{4-21}$$

$$\sum_{j:a_{ij}\in E} x_{ij}^{rs} - \sum_{j:a_{ji}\in E} x_{ji}^{rs} = \begin{cases} 1 & \forall i = r \\ 0 & \forall i \neq r, i \neq s \\ -1 & \forall i = s \end{cases} \qquad \forall r,s \in V \tag{4-22}$$

$$x_{ij}^{rs} \in \{0,1\} \quad \forall a_{ij} \in E, r \in V, s \in V \tag{4-23}$$

$$h \geq 0 \tag{4-24}$$

Mixed integer nonlinear programs (MINLPs) are difficult problems to solve and often require intelligent heuristics [Jiang and Adeli, 2003; Ferguson et al., 2012] and solution methods [Hajibabai and Ouyang, 2013]. Model P2 due to the conic constraint (21) falls into category of a class of mixed integer nonlinear programs known as MICQP. With recent development in solving the CQP's CPLEX and MOSEK have implemented branch and cut algorithm in order to solve MICQP efficiently. However the efficiency of solvers might go weak with increase in problem size. Therefore, in this dissertation an efficient outer approximation algorithm has been developed in order to solve the MICQP model P2 and demonstrated the computational advantages of the proposed solution framework over solving model P2 directly through the CPLEX solver.

4.3 Solution Methodology

Outer approximation has been customized and developed to solve the modeling formulation proposed for the robust shortest path problem. Before proceeding to the algorithm details the convexity of the formulation must be proved. To achieve this goal it is only suffice to prove the nonlinear function associate with the formulation is convex:

Lemma 1

$f(t_k, h) = \sqrt{\sum_k^K {t_k}^2} - h$ *is a convex function.*

Proof: Given the vector $T = (t_1, t_2, .. t_k)$, we can rewrite the function $f(t_k, h) = \|T\| - h,$ where $\|.\|$ is the Euclidean norm. Since *h* is a linear function we only need to prove that the term $\|T\|$ is convex.

Consider two arbitrary vectors, T_1 and T_2:

$\|(\lambda T_1 + (1-\lambda)T_2)\| = \|\lambda T_1 + (1-\lambda)T_2\|$

From the triangle inequality property in the norms we can conclude that:

$\|\lambda T_1 + (1-\lambda)T_2\| \leq \|\lambda T_1\| + \|(1-\lambda)T_2\|$

As the result we can write the inequality below:

$\|(\lambda T_1 + (1-\lambda)T_2)\| \leq \lambda\ \|T_1\| + (1-\lambda)\ \|T_2\|$

Therefore, the proof is complete.

4.3.1 Outer Approximation (OA) Subproblem(SP)

As previously discussed, the OA subproblem is formed by temporarily fixing all integer variables and then solving the resulting NLP. The goal of the OA subproblem is to find the optimal value for continuous variables t_k^l and h^l and approximate the nonlinear constraint (4-21) based on the achieved feasible point. It should be noted that in OA, the subproblem is usually the bottleneck for the algorithm as we need to solve one NLP at each iteration in order to obtain the optimal values for the continuous variables. However, in the case of model (P2) the optimal value for continuous variables t_k^l and h^l can be calculated through closed form equations and there is no need to solve the NLP for the SP as the solution is achieved trivially. In particular considering the feasible integer solutions ${\hat{x}_{ij}^{rs}}^l$ at every iteration $l \in L$, where L is the total number of the iterations, the optimal value for continuous variables $\hat{t}_k^l$ and $\hat{h}^l$ are calculated through equations (4-25) and (4-26) respectively:

$$\hat{t}_k^l = \sum_{(i,j)\in E} \sum_{(r,s)} b_{ijk} {\hat{x}_{ij}^{rs}}^l \qquad k \in K \qquad (4\text{-}25)$$

$$\hat{h}^l = \sqrt{\sum_{k \in K} {\hat{t}_k^l}^2} \qquad (4\text{-}26)$$

As the result the OA upper bound at every iteration l is

$$Z^l = \sum_{(i,j)\in E} \sum_{(r,s)} \bar{c}_{ij} {\hat{x}_{ij}^{rs}}^l + \Omega\, \hat{h}^l \qquad (4\text{-}27)$$

Starting the OA algorithm requires an integer feasible solution as the initial point. In the case of robust shortest path solving a classical shortest path problem will provide us an initial integer feasible solution to begin the algorithm.

4.3.2 Outer Approximation (OA) Master Problem (MP)

Given the optimal values for $\hat{t}_k^l$ and $\hat{h}^l$ at every iteration *l*, the OA MP is constructed by substituting equation (4-21) by its linear approximation through an iterative process. Theorem 1 describes the details on how to generate linear approximation of the constraint (4-21) at each iteration *l*.

Theorem 1

Let $\hat{t}_k^l$ and $\hat{h}^l$ denote the optimal solution for the nonlinear subproblem of the OA algorithm at iteration l, equation (4-28) is a valid linear approximation of the constraint (4-21).

$$\sum_{k\in K} \hat{t}_k^l(t_k - \hat{t}_k^l) - \hat{h}^l(h - \hat{h}^l) \leq 0 \tag{4-28}$$

Proof:

Considering $\hat{t}_k^l$ and $\hat{h}^l$as the optimal solutions of the SP, and taking into account the convexity the function $f(t_k, h)$, we can write the following equation according to the subgradient properties of the convex function:

$$f\left(\hat{t}_k^l,\ \hat{h}^l\right)+\nabla f\left(\hat{t}_k^l,\ \hat{h}^l\right)^{\mathrm{t}}\begin{bmatrix} t_k-\hat{t}_k^l \\ h-\hat{h}^l \end{bmatrix} \leq f(t_k,h) \leq 0 \qquad (4\text{-}29)$$

Thus, considering the convex function $f(t_k,h)=\sqrt{\sum_k {t_k}^2}-h$, we can expand equation (29) as below:

$$\sqrt{\sum_k \left(\hat{t}_k^l\right)^2}-\ \hat{h}^l+\frac{\sum_{k\in K}\hat{t}_k^l}{\sqrt{\sum_k\left(\hat{t}_k^l\right)^2}}\left(t_k-\hat{t}_k^l\right)-\left(h-\hat{h}^l\right)\leq 0 \qquad (4\text{-}30)$$

From the SP solution we have already calculated that $\hat{h}^l=\sqrt{\sum_{k\in K}{t_k^l}^2}$, therefore by considering equation (26) expression (30) can further be simplified into equation (31).

$$\sum_{k\in K}\frac{\hat{t}_k^l}{\hat{h}^l}\left(t_k-\hat{t}_k^l\right)-\left(h-\hat{h}^l\right)\leq 0 \qquad (4\text{-}31)$$

Moreover, assuming $\hat{h}^l\neq 0$ and premultiplying equation (31) by $\hat{h}^l$ the following equation is achieved and the proof is complete.

$$\sum_{k\in K}\hat{t}_k^l\left(t_k-\hat{t}_k^l\right)-\ \hat{h}^l\left(h-\hat{h}^l\right)\ \leq 0 \qquad (4\text{-}32)$$

Therefore the OA MP can be formulated as the following MILP:

OA MP

$$Z = \text{Min} \sum_{(i,j)\in E} \sum_{(r,s)} \bar{c}_{ij} x_{ij}^{rs} + \Omega h \tag{4-33}$$

s.t.:

$$\sum_{(i,j)\in E} \sum_{(r,s)} b_{ijk} x_{ij}^{rs} \le t_k \qquad \forall k \in K \qquad 4\text{-}34)$$

$$\sum_{k\in K} \hat{t}_k^l (t_k - \hat{t}_k^l) - \hat{h}^l \left(h - \hat{h}^l \right) \le 0 \qquad \forall l \in L \qquad 4\text{-}35)$$

$$\sum_{j:a_{ij}\in E} x_{ij}^{rs} - \sum_{j:a_{ji}\in E} x_{ji}^{rs} = \begin{cases} 1 \ \forall i = r \\ 0 \ \forall i \ne r, i \ne s \\ -1 \forall i = s \end{cases} \qquad \forall r, s \in V \tag{4-36}$$

$$x_{ij}^{rs} \in \{0,1\} \quad \forall a_{ij} \in E, r \in V, s \in V \tag{4-37}$$

$$h \ge 0 \tag{4-38}$$

The above formulation OA MP formulation is the MILP formulation for the OA Master Problem where at every iteration l a constraint of the form (4-35) is added to the problem to linearly approximate the convex nonlinear feasible region of model P2. The addition of constraint (4-35) at every iteration to the MP of the OA algorithm will provide a better approximation of the feasible region and thus result in non-decreasing sequence of the lower bounds. As mentioned before the cycling between the SP and MP is terminated when the predefined convergence measurement is reached. A complete detail of the algorithm steps is outlined in Table 4-1:

Table 4-1-Outer Approximation Steps

Outer Approximation (OA) Algorithm for Robust Shortest Path Problem
Input: Convergence tolerance ε, Maximum Number of Iteration L,
Upperbound(UB) = $+\infty$ Lowerbound (LB) =-∞, l=*1*
Initialization: Solve the classical shortest path problem for initial value of $\hat{\boldsymbol{x}}_{\boldsymbol{ij}}^{\boldsymbol{rs}^1}$
1: If (UB-LB) $\leq$ ε or $\boldsymbol{l} > L$ then go to step 7
2: Calculate the continuous variables $\hat{\boldsymbol{t}}_{\boldsymbol{k}}^{\boldsymbol{l}}$ and $\widehat{\boldsymbol{h}}^{\boldsymbol{l}}$ according to equations (4-25) and (4-26)
3: Calculate the subproblem objective function $\boldsymbol{Z}^{\boldsymbol{l}}$ according to equation (4-27)
4: If ($\boldsymbol{Z}^{\boldsymbol{l}} <$ UP), Update the upperbound and,
update the current best points as $\overline{\boldsymbol{x}}_{\boldsymbol{ij}}^{\boldsymbol{rs}} = \hat{\boldsymbol{x}}_{\boldsymbol{lk}}^{\boldsymbol{rs}^{\boldsymbol{l}}}$, $\bar{\boldsymbol{t}}_{\boldsymbol{k}} = \hat{\boldsymbol{t}}_{\boldsymbol{k}}^{\boldsymbol{l}}$ and $\overline{\boldsymbol{h}} = \widehat{\boldsymbol{h}}^{\boldsymbol{l}}$
5: Add equation (32) for iteration *l* and solve the MP based on equation (4-33) to (4-38)
6: Update the LB, *l*=*l*+*1*, go to step 1
7:Report $\overline{\boldsymbol{x}}_{\boldsymbol{ij}}^{\boldsymbol{rs}}$ and $\hat{\boldsymbol{t}}_{\boldsymbol{k}}^{\boldsymbol{l}}$ and $\overline{\boldsymbol{h}}$, Algorithm Stops

4.4 Numerical Experiment

In this section several numerical experiments are conducted in order to evaluate the quality of the solution achieved by the proposed formulation for robust shortest path problem. In particular, we demonstrate how accounting for different uncertainty levels with corresponding adjustments on protection levels would yield a shortest path decision different from the deterministic setting. We start the computational study using Sioux Falls network as a demonstration example and then extend the results for five medium to large networks. The network examples are selected from Bar-Gera's (2013) website based on the best known link flow solutions. More details on the size of the test networks are provided in Table 4-2. We also compare the efficiency of the proposed outer approximation algorithm with MICQP formulations solved by the CPLEX. Both OA algorithm and MICQP formulation for the robust shortest path problem are implemented in GAMS platform and solved by CPLEX 12.4 on a Dell Studio PC with Intel Core i5 processor running at 3.3 GHz and 8 GB of RAM under a 64 bit windows operating system.

Table 4-2. Test Networks

Networks	Number of Links(A)	Number of Nodes (N)
Sioux Falls	76	24
Anaheim	914	416
Barcelona	2522	1020
Chicago Sketch	2250	933
Chicago Regional	39,018	12292
Philadelphia	40,003	13,389

As discussed earlier, the uncertain travel cost is expressed as the combination of the nominal travel cost and a number independent random variables with associated weight b_{ijk} which represents the worst case effect of each of the uncertainty source on the uncertain link travel cost. For instance, assuming that four uncertainty sources of weather, accidents, road closure and transit failure as the sources of travel time uncertainty on a network. Also, considering that each one of the uncertainty sources can increase the travel cost up to 40%, 50%, 30% and 20% of the nominal state. Thus, according to equation (4-1) the link travel cost can be written based on the following relation:

$$\tilde{c}_{ij} = \bar{C}_{ij} + 0.4\bar{C}_{ij}y_{ij\,weather} + 0.5\bar{C}_{ij}y_{ij\,accident}$$
$$+0.3\bar{C}_{ij}y_{ij\,road\,closure}\,0.2\bar{C}_{ij}y_{ij\,transit\,failure}$$

Therefore, in order to empirically calculate the weight parameter we only need know the worst case effect of each uncertainty source on link travel cost rather that the knowledge of the whole distribution of the uncertainty source. In the first part of the experiment we consider two cases; In the first case we assume that there are four sources of uncertainty with associated weight $b_{ijk} = \delta\bar{c}_{ij}$ where $\delta \in \{0.2, 0.4, 0.6, 0.8\}$. In the second part of the experiment weights are randomly generated from a uniform distribution with range $[-\bar{c}_{ij}, \bar{c}_{ij}]$. Also we should point out that in this particular test experiment weights b_{ijk} are generated in such a way that the final link travel costs are all positive. As explained earlier these uncertainty sources could correspond to network wide effects such as weather or local effects such as an accident. A transportation engineer may potentially estimate these weights by studying data on the variation in link travel costs and using a combination of statistical analysis and engineering judgment to arrive at the weights of the different sources of uncertainty. The focus of this work is the mathematical formulation and solution algorithm of the robust shortest path problem when uncertain cost is characterized by a known mean and affine combination of several independent sources of uncertainty. Estimating the weights from real world data is beyond the scope of this work.

5.1 Experiment 1:

We first consider Sioux Falls as a simple small network to study the applicability of the robust optimization approach in finding the shortest path decisions. Generally, the quality of the robust solution is evaluated as the tradeoff of between the cost of robust and the deterministic solutions. Toward this end, defining x_{ij}^{rs} as the solution of the deterministic shortest path and ${x_{ij}^{rs}}^*$ as the solution of the robust model the robust solution is evaluated based on the ratio of cost of a robust path to the deterministic solution.

$$\text{Relative cost}= \frac{\sum_{(i,j)\in E} \bar{c}_{ij} {x_{ij}^{rs}}^*}{\sum_{(i,j)\in E} \bar{c}_{ij} x_{ij}^{rs}}$$

This relative cost shows the ratio of the cost with respect to the deterministic setting that is incurred to the network users in order to hedge against the realized uncertainty. Increase in the relative cost ratio implies that higher margin of cost is required in order to deal with the existing uncertainty.

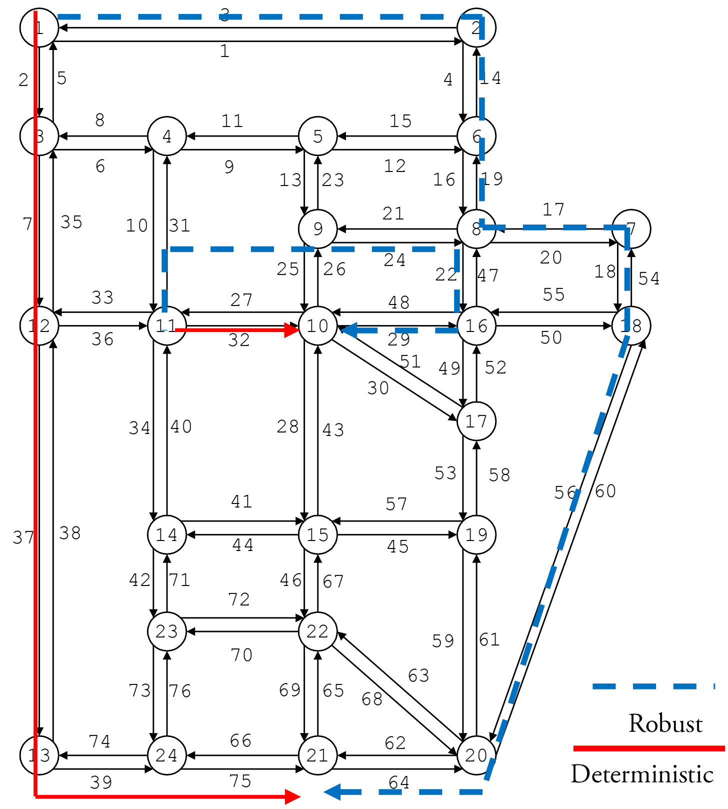


Figure 4-1: Robust vs deterministic shortest path

The depicted results in Figures 4-2 and 4-3 imply that the cost of the robust path would increase with corresponding increase in uncertainty budget. This observation can be attributed to the fact that increase in uncertainty budget would increase the protection level which as a result increases the cost of the robust path. We should point out that the cost of the shortest path in a robust optimization framework is greater or at least equal to the cost of the deterministic shortest path since the traditional determinist solution is a path with the least travel cost; therefore the defined relative cost ratio is always greater or equal to one. This is sometimes stated as the sub-optimality of the robust solution in compare to the deterministic programming. However, given the uncertainty budget robust optimization is seeking for a solution with minimal sub-optimality compared to the deterministic problem. For example in this experiment the link travel costs uncertainty level of δ =0.8 is translated as up to 80% perturbation in link travel costs whereas the cost of the robust shortest path for O-D pair 1-23 seemed to be only 17.7 % higher in compare to the deterministic shortest path solutions. This means that increase of 17.7% in total routing cost can protect the path decision against 80% of increase in travel cost. Similar results were achieved for O-D path (12-22) where the cost of the robust path was 33% higher than the deterministic shortest path across all the uncertainty levels. We also observe that for the considered O-D pairs with increase in uncertainty budget the relative cost of the robust shortest path compared with the deterministic case for all the considered values of δ would converge to an upperbound cost. Therefore in these cases one path tends to be robust regardless of considered uncertainty level δ. In addition Table 4-3 provides the details of the robust and deterministic path along with the cost comparison for several selected O-D pairs. Graphical representation of two O-D pairs (OD pairs (1,21) and (10,16)) has been demonstrated in Figure 4-1 for Sioux Falls network.

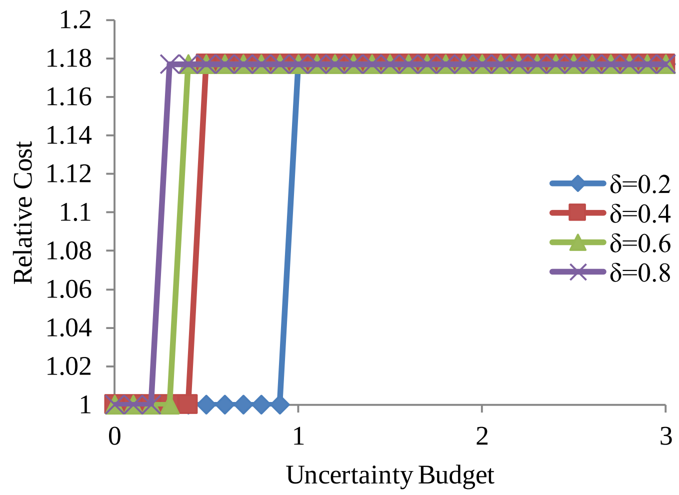


Figure 4-2-Sioux Falls Network. O-D pair (1-23)

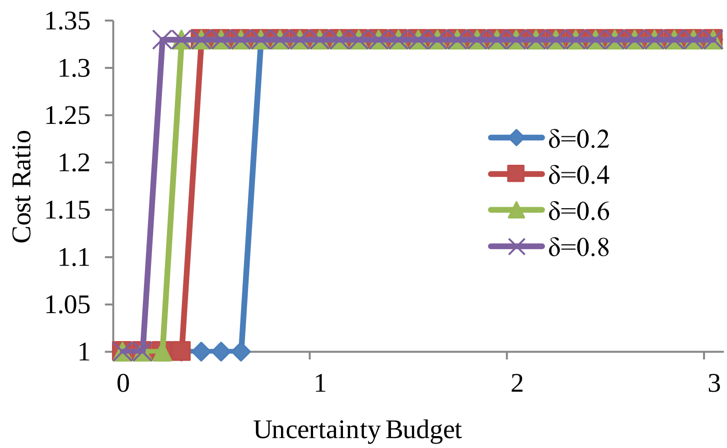


Figure 4-3- Sioux Falls Network. O-D pair (12-22)

Table 4-3. Deterministic and robust paths for 10 O-D pairs of Sioux Falls

O-D Pair	Deterministic Path	Robust Path	Relative Cost
(1-23)	1-3-12-13-24-23	1-3-4-11-14-23	1.177
(1-21)	1-3-12-13-24-21	1-2-6-8-7-18-20-21	1.168
(3-22)	3-12-13-24-23-22	3-4-5-9-10-15-22	1.1
(10-16)	10-16	10-9-8-7-18-16	1.568
(11-12)	11-12	11-4-3-12	1.142
(4-21)	4-3-12-13-24-21	4-5-9-10-15-22-21	1.097
(9-11)	9-10-11	9-5-4-11	1.067
(12-22)	12-13-24-23-22	12-3-4-5-9-10-15-22-	1.33
(6-17)	6-8-7-18-16-17	6-8-7-18-16-17	1
(10-12)	10-11-12	10-9-5-4-3-12	1

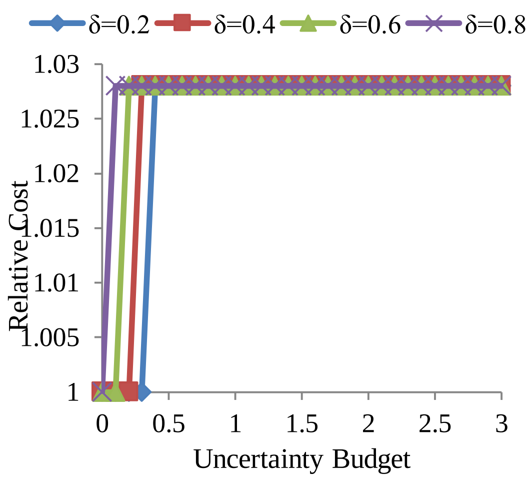
We continue our experiments for Sioux Falls by calculating the robust shortest path for ten randomly selected O-D pairs in order to illustrate the difference in robust and the basic deterministic path in terms of the arcs forming the shortest path. Similar to the previous experiment we also compare the robust path cost with the deterministic base case. We considered uncertainty budget $\Omega = 3$ and assumed that $\delta = 0.4$. As expected, the results indicated a different path decision in the presence of uncertainty with relatively higher cost compared to the base case. However for two O-D pairs robust and the deterministic paths have the same cost. As we can see from the results for O-D pair (6-17) shortest path coincide with the deterministic path. This shows the fact that in some cases one path can be optimal regardless of the realized uncertainty. Also, the robust and the deterministic path for O-D pair (10-12) have the same cost while being different in the arcs comprising the path.

The sensitivity of the robust shortest path decision with respect to uncertainty level and uncertainty budget is also tested on five more medium to large size networks. We calculated the shortest path on two selected O-D pairs for every network. Similar to the previous experiments the quality of the solution is presented according to the relative cost of the robust shortest path with respect to deterministic shortest path for different uncertainty levels and uncertainty budget. The results of this experiment are illustrated in Figures (4-4, 4-5, 4-6, 4-7 and 4-8). Similar to Sioux Falls case, the results confirm that the cost of the robust shortest path is higher than the deterministic one. As expected, the general trend for all the network is that the cost of the robust path increase with increase in uncertainty budget since higher uncertainty budget provide more protected solutions with higher cost. Furthermore, we can observe from the results that the robust shortest path tends to be more sensitive to the realized uncertainty levels and uncertainty budget considerations with increase in network size. In particular, the robust shortest path seems to be the same across all uncertainty levels and budgets for networks such as Anaheim, Barcelona, and Chicago Sketch whereas for larger networks such as Chicago Regional and Philadelphia robust shortest path varies with respect to uncertainty and level and budget assumptions. This is mainly due to the fact that total number of paths between O-D pairs tends to increase significantly with increase in network size thus providing more flexibility to choose among various available paths.

Hence frequency of changing the robust shortest path with increase in uncertainty budget seems to be more for larger networks. Also, the cost of the robust path for smaller networks would converge to an upper limit cost with increase in the uncertainty budget (Figures 4-2,4-3,4-4, 4-5 and 4-6) whereas in larger networks such as Chicago Regional (Figure 4-8) and Philadelphia (Figure 4-9) due to the large number of available path between the O-D pairs this may not be the case.

A different set of experiment is also designed to investigate a general case where ten instances of the robust shortest path in which $b_{ijk} = uniform(-1,1)\bar{c}_{ij}$ are solved. In addition, we should mention that in this particular test experiment weights b_{ijk} are generated in such a way that the final link travel costs are all positive. Also, in order to examine the effect of uncertainty budget three various scenarios of Ω=1, 2 and 3 are considered. Moreover in this test experiments we considered ten uncertainty sources affecting the link travel costs. The results of this experiment have been plotted in Figure 9 where the average relative cost of the ten randomly generated scenarios with respect to the base deterministic shortest path has been calculated. As expected the results show that with increase in uncertainty budget the cost of the robust shortest path would increase since higher uncertainty budget would require solutions which are more robust to uncertainty and thus the cost would increase.

To further investigate the effect of robust optimization related parameter on the OA algorithm, an additional experiment step has been devised which study the impact of number of factors on the solution time of the OA algorithm. Toward this goal, we considered three cases where 4, 10, and 25 factors contributing to travel cost uncertainty are assumed for calculating the robust shortest path. The weight for each factor has been randomly generated by the equation $b_{ijk} = uniform(-1,1)\bar{c}_{ij}$ while making sure the final link travel cost is positive. We generated ten randomly scenarios and solve the robust shortest path for the test networks based on Ω=3. In addition, we set the rest of the parameters based on the previous experiment. The minimum, average and the maximum of computational effort for each of the networks has been displayed in Table 4-5. The presented results show that computational time grows with increase in number of factors. This is because with increase in number of factors not only increases the size of the formulation but also the nonlinear constraints (21) and their associated linear constraints would increase in size which as the result makes the problem more difficult to solve. Furthermore, the increase in computational time tends to be more significant in larger networks such as Chicago Regional and Philadelphia due to the size effects.



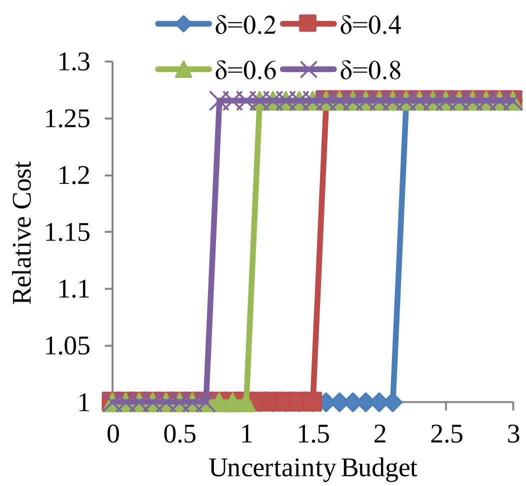


Figure 4-4 Anaheim. Left Figure: O-D pair (1-400), Right Figure O-D pair (11-326)

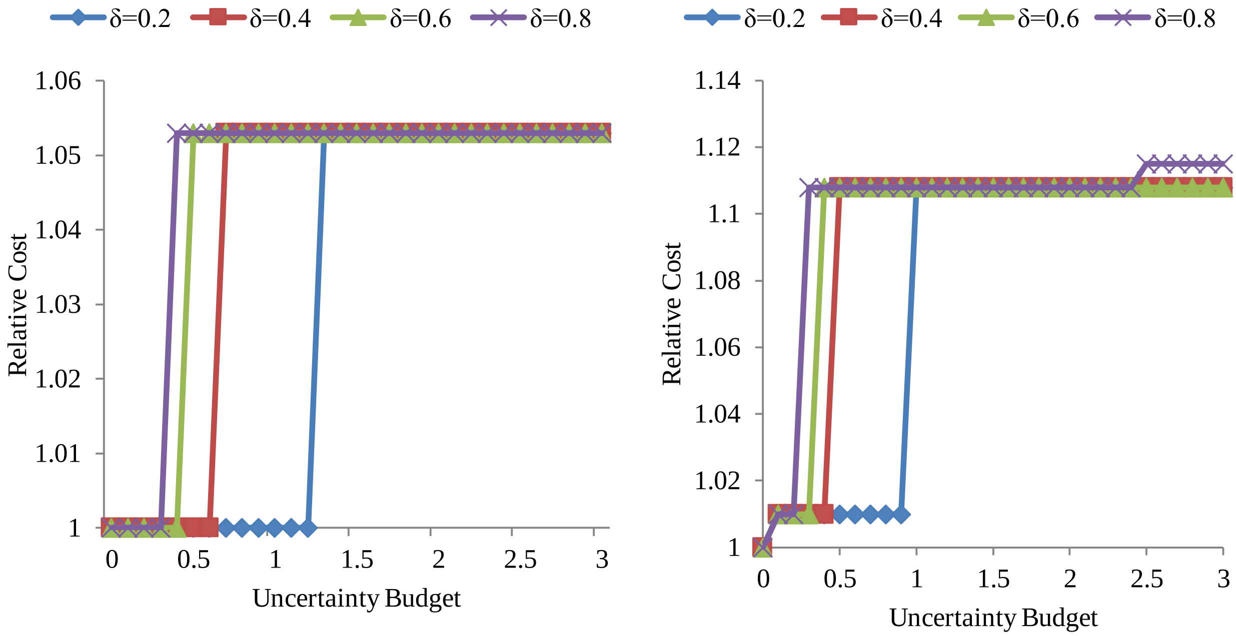


Figure 4-5 Chicago Sketch. Left Figure: O-D pair (1-933), Right Figure O-D pair (28-584).

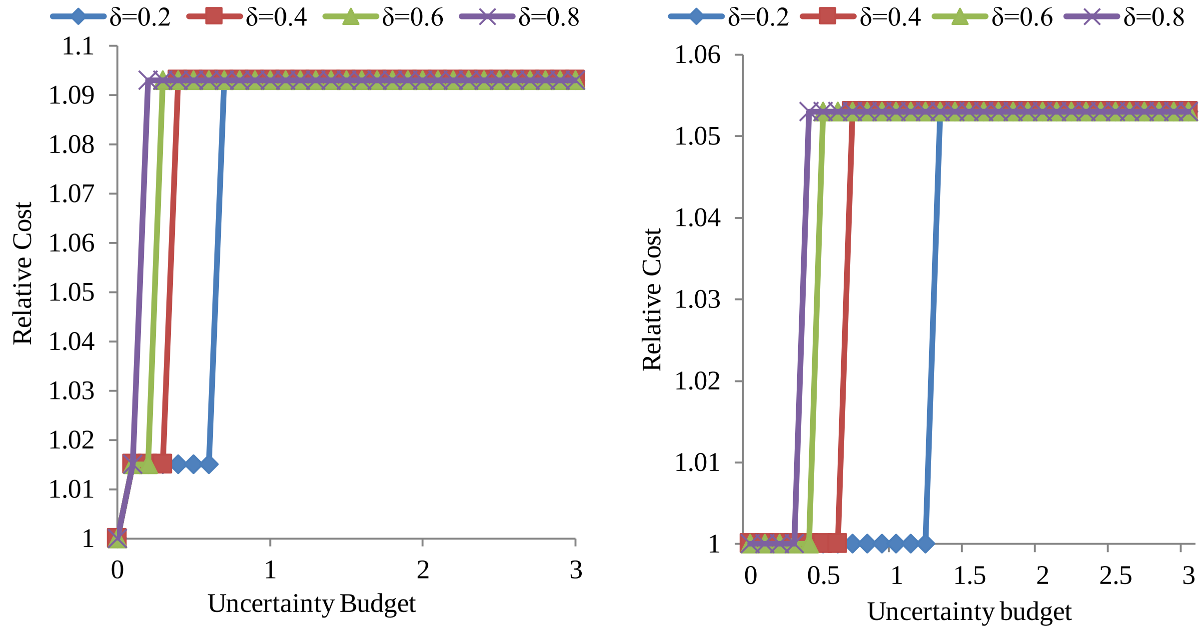


Figure 4-6 Barcelona. Left Figure: O-D pair (1-1020), Right Figure O-D pair (235-1004).

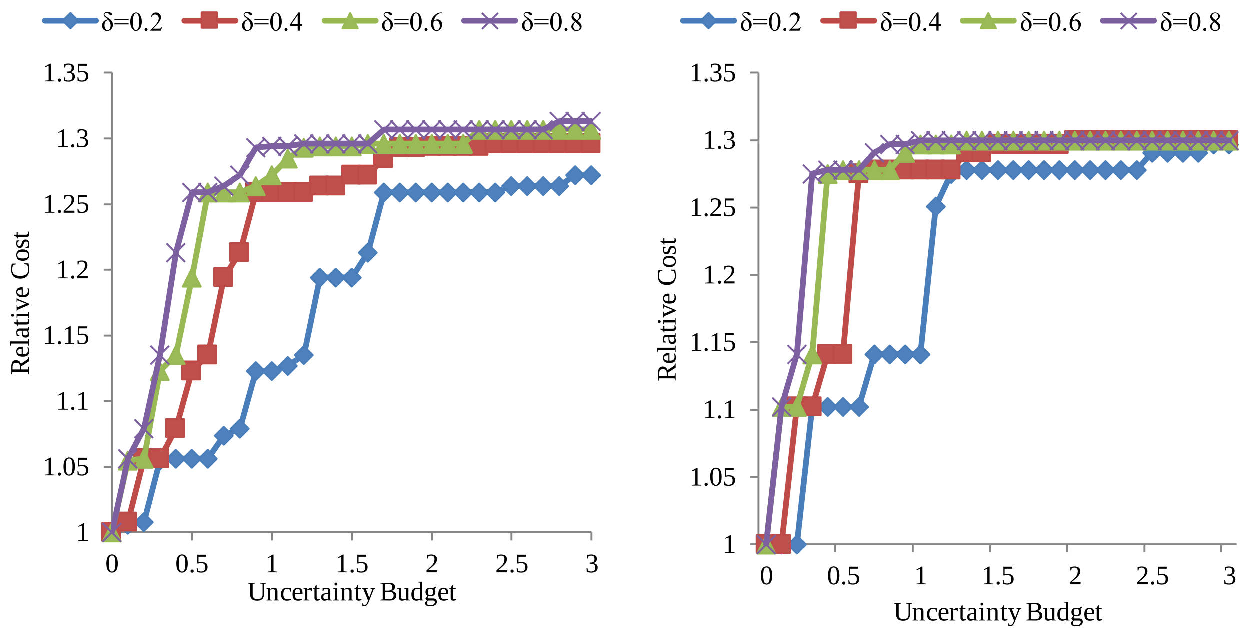


Figure 4-7 Chicago Regional. Left Figure: O-D pair (1234-5587), Right Figure O-D pair (1-12982).

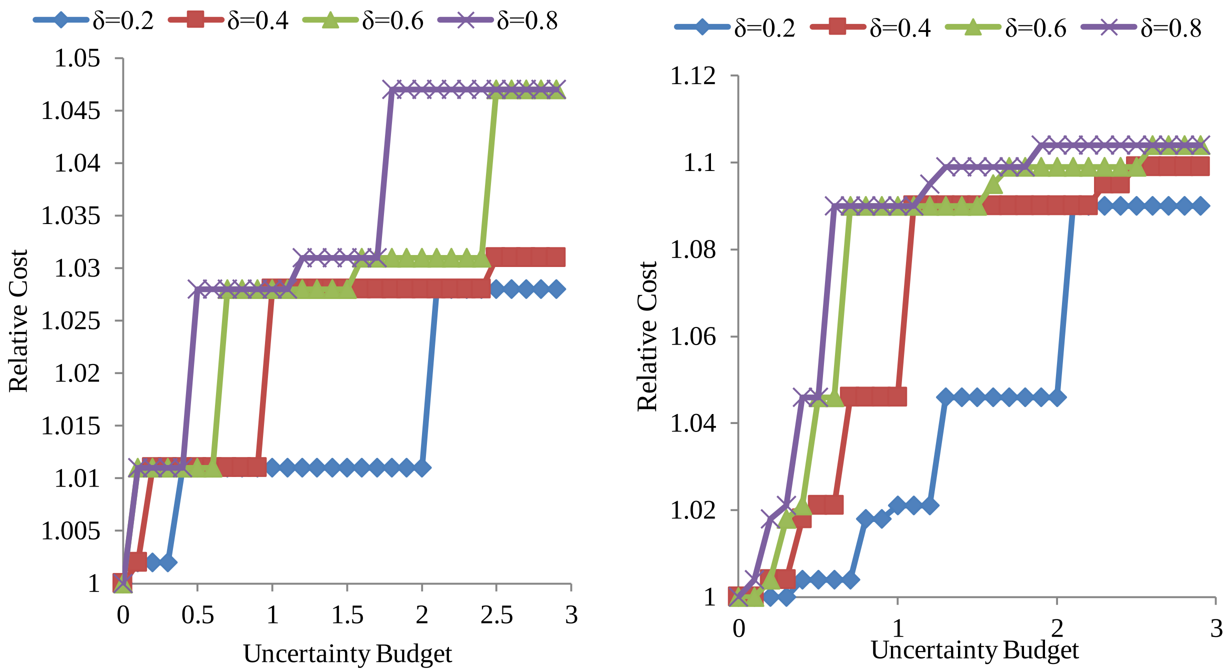


Figure 4-8 Philadelphia. Left Figure: O-D pair (1-13123), Right Figure O-D pair (2437-10967).

Table 4-4-Computational time versus number of uncertainty factors

Network	Number of Factors								
	4			10			25		
	Computational Time (sec)								
	Min	Ave	Max	Min	Ave	Max	Min	Ave	Max
Sioux Falls	0.068	0.077	0.081	0.62	0.081	0.115	0.078	0.098	0.125
Anaheim	0.156	0.173	0.202	0.182	0.265	0.342	0.246	0.378	0.515
Barcelona	0.266	0.385	0.437	0.458	0.884	1.48	0.701	1.583	2.053
Chicago Sketch	0.639	0.786	0.887	0.498	1.247	1.592	0.998	1.563	1.886
Chicago Regional	2.682	5.149	7.962	3.014	5.865	8.231	3.251	7.628	9.362
Philadelphia	2.852	5.492	9.365	6.025	12.011	26.387	16.38	29.53	48.892

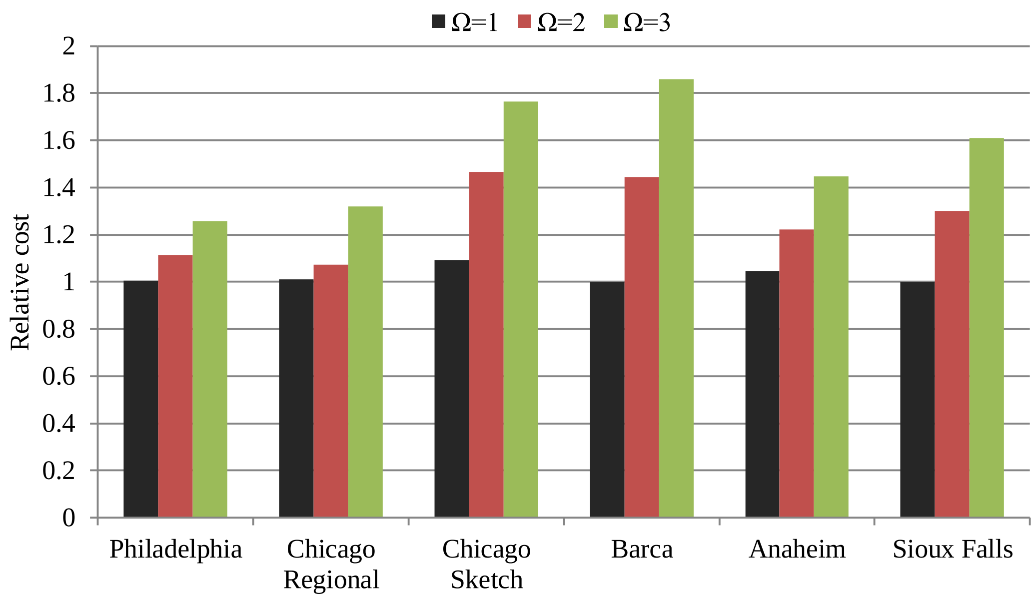


Figure 4-9 Robust path cost versus uncertainty budget

Table 4-5- Comparison results of CPLEX solver versus the OA algorithm- $\boldsymbol{\delta = 0.4}$

Network	CPLEX			OA-Algorithm				
	Running time per path (Sec)			Running time per path (Sec)			Iteration Number	
	Min	Ave	Max	Min	Ave	Max	Min	Ave
Sioux Falls	0.031	0.038	0.115	0.004	0.032	0.082	1	2
Anaheim	0.031	0.059	0.131	0.008	0.043	0.091	1	3
Barcelona	0.051	0.142	0.263	0.018	0.075	0.182	1	3
Chicago Sketch	0.031	0.081	0.187	0.015	0.066	0.141	1	3
Chicago Regional	1.669	2.281	4.102	0.390	0.656	2.434	2	5
Philadelphia	1.139	1.485	2.623	0.375	0.669	1.763	2	5

Table 4-6 Comparison results of CPLEX solver versus the OA algorithm- $\delta = 0.8$

Network	CPLEX			OA-Algorithm				
	Running time per path (Sec)			Running time per path (Sec)			Number of Iteration	
	Min	Ave	Max	Min	Ave	Max	Min	Ave
Sioux Falls	0.034	0.042	0.183	0.014	0.041	0.102	1	2
Anaheim	0.042	0.071	0.169	0.038	0.063	0.141	1	3
Barcelona	0.063	0.142	0.281	0.039	0.091	0.212	1	3
Chicago Sketch	0.061	0.111	0.234	0.045	0.116	0.197	1	3
Chicago Regional	1.822	2.624	4.805	0.401	0.685	2.902	2	5
Philadelphia	1.685	2.130	3.915	0.405	0.692	3.12	2	5

4.4.1 Experiment 2

In this section of the numerical experiment the efficiency of the proposed OA algorithm for the robust shortest path problem is investigated by comparing the solution time of the OA algorithm with the direct implementation of model P2 as a MICQP program. CPLEX solver was selected for solving the MICQP program and formulation was coded in GAMS platform. We considered uncertainty budget $\Omega = 3$ and assumed that $\delta = 0.4$ and $\delta = 0.8$. In the entire tests carried out in this part node (1) was selected as the origin node and the robust shortest path was calculated across all destinations except for Chicago Regional and Philadelphia networks where only 1500 random destination nodes were selected. Minimum, maximum, and average running time required by the two methods for delivering the desired solution as well as the minimum and maximum number of OA iterations are reported as the performance measure in order to evaluate their efficiency of the algorithm. The optimality gap was set at 0 percent for CPLEX solver and the same value was selected as the OA stopping criteria. Moreover, the OA MP was solved to optimality. Figure 10 illustrates the convergence of OA approach for Philadelphia network where the trajectory of the algorithm till convergence has been presented. More specifically the convergence has been plotted based on the solution of the equation (27)-upperbound and OA-MP (lowerbound) respectively. The results show that for this considered instance OA converges in three iterations of the algorithm.

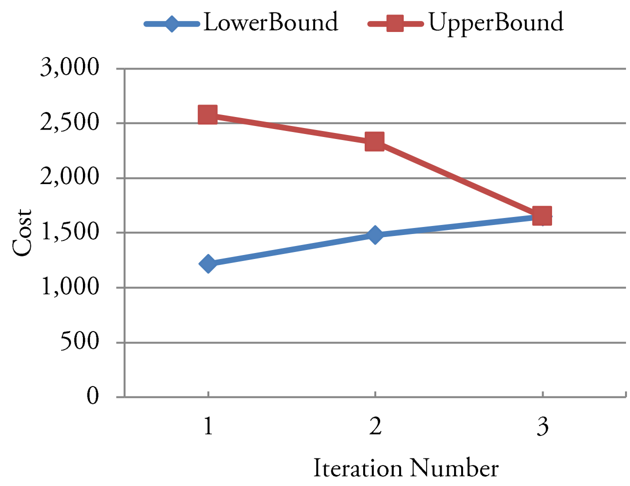


Figure 4-10 Convergence of the OA algorithm for Philadelphia network

In addition, the results displayed in Table 5 show that OA requires less time in solving the MICQP formulation P2 when compared to the direct implementation of formulation P2 as a MICQP in CPLEX solver. As expected this difference in computational time tends to increase for the larger networks which demonstrates the ability of the OA algorithm in handling large size networks. Moreover, the comparison of the reported results in Tables 5 and 6 shows

that the solution time for both methods is independent of the uncertainty level δ as the solution time of the two approaches for $\delta = 0.4$ and $\delta = 0.8$ seemed to be very close. The results also show that OA was able to deliver optimal solution in maximum of five iterations for the largest networks considered in this study. In addition, while we did not conduct a detailed computational complexity analysis, empirically the computational time was found to increase linearly with the number of nodes and the number of uncertainty factors in the network.

Chapter 5 Non-additive Shortest Path Problem

5.1 Introduction

The shortest path problem is a fundamental network flow problem with applications in many domains such as transportation, telecommunication, logistics etc. The shortest path problem involves determining the least cost path comprised of several arcs connecting an origin node to a destination node. A significant amount of research has been devoted to studying the shortest path assuming the additive link travel costs which allows for the application of highly efficient dynamic programming algorithms based on Bellman principal of optimality. However in several applications path costs may not always be an additive function of link travel costs [McCord, Villoria, 2007].

Several researchers have noted the importance of capturing traveler's nonlinear preference and non-additive preferences in travel disutility. For example Pinjari and Bhat [2006] experiments on commute travel mode choice of Austin Commuter Stated Preference Survey data showed that not accounting for nonlinearities in travel time and travel time unreliability would highly affect the validity of the model and results in unrealistic and inconsistent outcomes. Therefore for modeling point of view consideration of nonlinear preferences in calculating the shortest path decision seems to be essential. Redmond and Mokhtarian [2001], De Palma and Picard [2005], Fosgerau and Karlstrom [2010] have developed and compared several specifications based on standard deviation, variance in order to capture traveler's degree of risk aversion. There is a need to develop shortest path algorithms which can capture the nonlinear and non-additive preferences of travelers to accurately capture the travelers routing behavior.

In this chapter the non-additive shortest path problem with multiple differentiable convex travel cost attribute functions is formulated as a convex mixed integer nonlinear program. An outer approximation based solution strategy is developed to solve the problem. The algorithm is proven to converge to optimality for the functional forms considered in this part.

5.2 Formulation

Let *G(V,E)* be a directed graph where *V* is the set of nodes and *E* is the set of directed links, and elements a_{ij} denote links connecting node *i* to node *j*. Also, let node $r \in V$ and node $s \in V$ represent the origin and destination node respectively. Consider x_{ij}^{rs} as the binary decision variable determining if link a_{ij} lies on the shortest path. Moreover, assume that we have *K* nonlinear path travel cost functions $U_k: \mathbb{R}^+ \to \mathbb{R}^+$ which are transforming the paths attributes into a representative disutility measure. In this chapter path travel cost functions are assumed to be positive convex differentiable. The non-additive shortest path problem in its general form is given as below:

P1:

$$Z = Min \sum_{k=1:K} U_k \left(\sum_{a_{ij} \in E} c_{ij} x_{ij}^{rs} \right) \tag{5-1}$$

$$\sum_{j:a_{ij} \in E} x_{ij}^{rs} - \sum_{j:a_{ji} \in E} x_{ji}^{rs} = \begin{cases} 1 & \forall\, i = r \\ 0 & \forall\, i \neq r, i \neq s \\ -1 & \forall\, i = s \end{cases} \tag{5-2}$$

$$x_{ij}^{rs} \in \{0,1\} \qquad \forall a_{ij} \in E, s \in S, r \in R \tag{5-3}$$

Equation (5-1) in the above formulation is the objective function which is the sum of various nonlinear path travel cost. Constraint (5-2) is the flow balance constraints representing the flow of node *i.* Equation (5-3) ensures that the decision variable x_{ij}^{rs} is binary. The goal of the formulation is to find the robust shortest path from origin node *r* to destination node *s* in order to minimize the sum of *k* non-additive convex nonlinear cost functions. It should be noted that the formulation P1 is a binary integer programming formulation which is convex in the objective function and linear in constraints. In order to further exploit the properties of formulation we introduce auxiliary positive variable t_k and transform the initial binary integer program P1 to mixed integer nonlinear program P2. The transformation of the primary formulation P1 to P2 allows for the application of common methods such as outer approximation (OA) [Duron and Grossman, 1986] and generalized benders decomposition (GBD) [Geoffrion, 1972] for solving MINLP. The MINLP counterpart of model P2 is presented as below:

P2

$$Z = Min \sum_{k} t_k \tag{5-4}$$

$$U_k\left(\sum_{a_{ij} \in E} c_{ij} x_{ij}^{rs}\right) \leq t_k \qquad \forall k \tag{5-5}$$

$$\sum_{j:a_{ij} \in E} x_{ij}^{rs} - \sum_{j:a_{ji} \in E} x_{ji}^{rs} = \begin{cases} 1 & \forall\, i = r \\ 0 & \forall\, i \neq r, i \neq s \\ -1 & \forall\, i = s \end{cases} \tag{5-6}$$

$$x_{ij}^{rs} \in \{0,1\}, t_k \geq 0 \qquad \forall a_{ij} \in E, s \in S, r \in R, \forall k \tag{5-7}$$

In the above formulation variable t_k represent the nonlinear travel cost in the objective function and therefore the transformed formulation is linear in objective function but nonlinear and convex in constraints. It should be noted that the convexity of formulation P2 is arising from convexity of continuous relaxation of travel cost function $U_k\left(\sum_{a_{ij} \in E} c_{ij} x_{ij}^{rs}\right)$ over positive real numbers. More specifically according to Bonami et al *(33)* since in the above formulation $U_k\colon \mathbb{R}^+ \to \mathbb{R}^+$ is convex and differentiable therefore MINLP model of P2 is also convex. In this chapter OA algorithm is adopted as a general method for solving MINLP formulation P2. More details on OA algorithm procedure has been presented in below:

5.2.1 OA Subproblem

In OA algorithm the role of the subproblem is to optimize the value for continuous variables. Recalling from the previous discussion the OA SP is a nonlinear formulation in which all the integer assignments are fixed. In particular, let ${x_{ij}^{rs}}^h$represent a feasible integer assignment at iteration *h,* SP is seeking to optimize the continuous variable ${t_k}^h$ at every iteration *h* by fixing the integer variables. In the OA framework solving the NLP SP can itself be a hard problem and usually forms the bottleneck for the algorithm, however in the case of non-additive shortest path formulation P2, due to the special structure of the problem the value of continuous variables can be determined by substitution of the integer assignment in constraint (5-5). In other words due to the structure of the problem, the value of continuous variable can be obtained from the integer assignments without solving the NLP. In particular, given the feasible integer assignment of ${x_{ij}^{rs}}^h$ at iteration *h* continuous variable ${t_k}^h$ is computed through closed form equation (5-8). As a result OA upper bound can be calculated through substitution of the optimal value of ${t_k}^h$ into the objective function as presented in equation (5-9).

$$t_k{}^h = U_k\left(\sum_{a_{ij} \in E} c_{ij} x_{ij}^{rs\,h}\right) \qquad \begin{array}{c} \forall k \\ \forall h = 1..H \end{array} \tag{5-8}$$

$$Z^h = Min \sum_k t_k{}^h \qquad \forall h = 1..H \tag{5-9}$$

Equation (5-9) which is the objective function of the SP provides the upper bound of the algorithm at very iteration.

5.2.2 OA Master Problem

Consider $g(x,t) = U_k\left(\sum_{a_{ij} \in E} c_{ij} x_{ij}^{rs}\right) - t_k$, since $U_k: \mathbb{R}^+ \to \mathbb{R}^+$ is a convex differentiable function and t_k is linear therefore $g(x)$ is also convex and differentiable. As already discussed the basic intuition behind the outer approximation strategy is to transform the original integer nonlinear problem into an equivalent integer linear problem by iteratively adding linearized approximation of the nonlinear terms to the linear relaxed program. The generation of linear approximation is based on the subgradient properties of the objective function. In particular, given ($x_{ij}^{rs\,h}$, $t_k{}^h$) as a feasible solution at iteration *h,* the supporting hyperplane for $g(x,t)$ can be written as below:

$$g\left(x_{ij}^{rs\,h}, t_k{}^h\right) + \nabla g\left(x_{ij}^{rs\,h}, t_k{}^h\right)^T \begin{bmatrix} x_{ij}^{rs} - x_{ij}^{rs\,h} \\ t_k - t_k{}^h \end{bmatrix} \leq g\left(x_{ij}^{rs}, t_k\right) \leq 0 \tag{5-10}$$

Substituting $g(x,t) = U_k\left(\sum_{a_{ij} \in E} c_{ij} x_{ij}^{rs}\right) - t_k$ in the above expression we will have the OA cuts as follows:

$$U_k\left(\sum_{a_{ij} \in E} c_{ij} x_{ij}^{rs\,h}\right) + \sum_{a_{ij} \in E} c_{ij}\ U_k{}'\left(\sum_{a_{ij} \in E} c_{ij} x_{ij}^{rs\,h}\right) \times \left(x_{ij}^{rs} - x_{ij}^{rs\,h}\right) - t_k{}^h - (t_k - t_k{}^h)\ \leq 0 \qquad \forall k = 1..K \quad \forall h = 1..H \tag{5-11}$$

In the above expression $U_k{}'\left(\sum_{a_{ij} \in E} c_{ij} x_{ij}^{rs\,h}\right)$ represents the first derivative of the cost function U_k. Moreover considering the equation (5-8) expression (5-10) can be further simplified to the following set of cut:

$$\sum_{a_{ij} \in E} c_{ij}\ U_k{}'\left(\sum_{a_{ij} \in E} c_{ij} x_{ij}^{rs\,h}\right) \times \left(x_{ij}^{rs} - x_{ij}^{rs\,h}\right) - (t_k - t_k{}^h)\ \leq 0 \qquad \forall k = 1..K \quad \forall h = 1..H \tag{5-12}$$

It should be noted that the approximation scheme presented above linearly underestimates the nonlinear objective functions and over estimates the feasible region. Therefore the OA MP can be presented as below:

OA MP:

$$LB = Min\ \sum_k t_k \tag{5-13}$$

$$\sum_{a_{ij} \in E} c_{ij}\ U_k{}'\left(\sum_{a_{ij} \in E} c_{ij} x_{ij}^{rs\,h}\right) \times \left(x_{ij}^{rs} - x_{ij}^{rs\,h}\right) - \qquad \forall k = 1..K \quad \forall h = 1..H \tag{5-14}$$

$$(t_k - {t_k}^h) \leq 0$$

$$\sum_{j:a_{ij} \in E} x_{ij}^{rs} - \sum_{j:a_{ji} \in E} x_{ji}^{rs} = \begin{cases} 1 & \forall\, i = r \\ 0 & \forall\, i \neq r, i \neq s \\ -1 & \forall\, i = s \end{cases} \qquad (5\text{-}15)$$

$$LB^{h-1} \leq \sum_k t_k \qquad \forall h = 1..H \qquad (5\text{-}16)$$

$$LB^h \leq UB \qquad \forall h = 1..H \qquad (5\text{-}17)$$

$$x_{ij}^{rs} \in \{0,1\} \qquad \forall a_{ij} \in E, s \in S, r \in R \qquad (5\text{-}18)$$

$$t_k \geq 0 \qquad \forall k \in K \qquad (5\text{-}19)$$

The above formulation is the linear approximation equivalent model for the original formulation (P2) in which at every iteration h of the OA algorithm a linear constraints in form of equation (5-14) for every nonlinear function U_k would be added to the formulation to approximate the convex feasible region. Due to the accumulation of OA cuts at each iteration *h* the master problem will generate a non-decreasing sequence of lower bounds. In our model, constraint (5-16) is also added to OA MP which guarantees that the lower bound generated at each iteration *h* is greater or equal to the lower bound achieved at previous iterations. Furthermore constraints (5-16) will improve the efficiency of the OA strategy by reducing the branch and bound three. Constraint (5-17) is added to the formulation to ensure that the solution of the MP is less than the upper bound. OA converges if the difference between the updated upper and lower bound satisfy the termination criteria. It should be noted that the OA algorithm requires an initial feasible solution in order to start the algorithm. In the case of non-additive shortest path problem a feasible path is considered as the starting initial point for the algorithm. We should note that the feasible path can be obtained by solving a traditional shortest path problem. More details on the algorithm steps are presented in below:

Step 0-Algorithm Set up and Initialization
Consider the ε as the convergence tolerance, H as the maximum number of iterations, Upperbound(UB) = $+\infty$ Lowerbound (LB) =-∞ and *h=1;*
Obtain a feasible path as the initial integer assignment $\hat{x}_{ij}^{rs^1}$
Step 1-Solve the continuous variables and updating the upper bound
Solve the continous variables $\hat{\boldsymbol{t}}_{\boldsymbol{k}}^{\boldsymbol{h}}$ according to equations (5-8), calculate the subproblem objective function $\boldsymbol{Z}^{\boldsymbol{h}}$ according to equation (5-9). Also, if ($\boldsymbol{Z}^{\boldsymbol{h}} <$ UP), update the upperbound and update the current best points as $\overline{\boldsymbol{x}}_{\boldsymbol{ij}}^{\boldsymbol{rs}} = \hat{\boldsymbol{x}}_{\boldsymbol{lk}}^{\boldsymbol{rs}^{\boldsymbol{h}}}$, $\bar{\boldsymbol{t}}_{\boldsymbol{k}} = \hat{\boldsymbol{t}}_{\boldsymbol{k}}^{\boldsymbol{h}}$.
Step 2- Solve the MP in order to optimize the integer variables
Add equation (5-14) for iteration *h* of the algrithm and solve the MP based on equation (5-13) to (5-19).
and update the LB, *h=h+1*, go to step 3.
Step 3-Convergence Check
If (UB-LB) ≤ ε or $\boldsymbol{h} > H$ then go to step 4 otherwise go to step 1.
Step 4-Termination of the algorithm and report of solution
Report $\overline{\boldsymbol{x}}_{\boldsymbol{ij}}^{\boldsymbol{rs}}$ and $\hat{\boldsymbol{t}}_{\boldsymbol{k}}^{\boldsymbol{h}}$ and the algorithm stops.

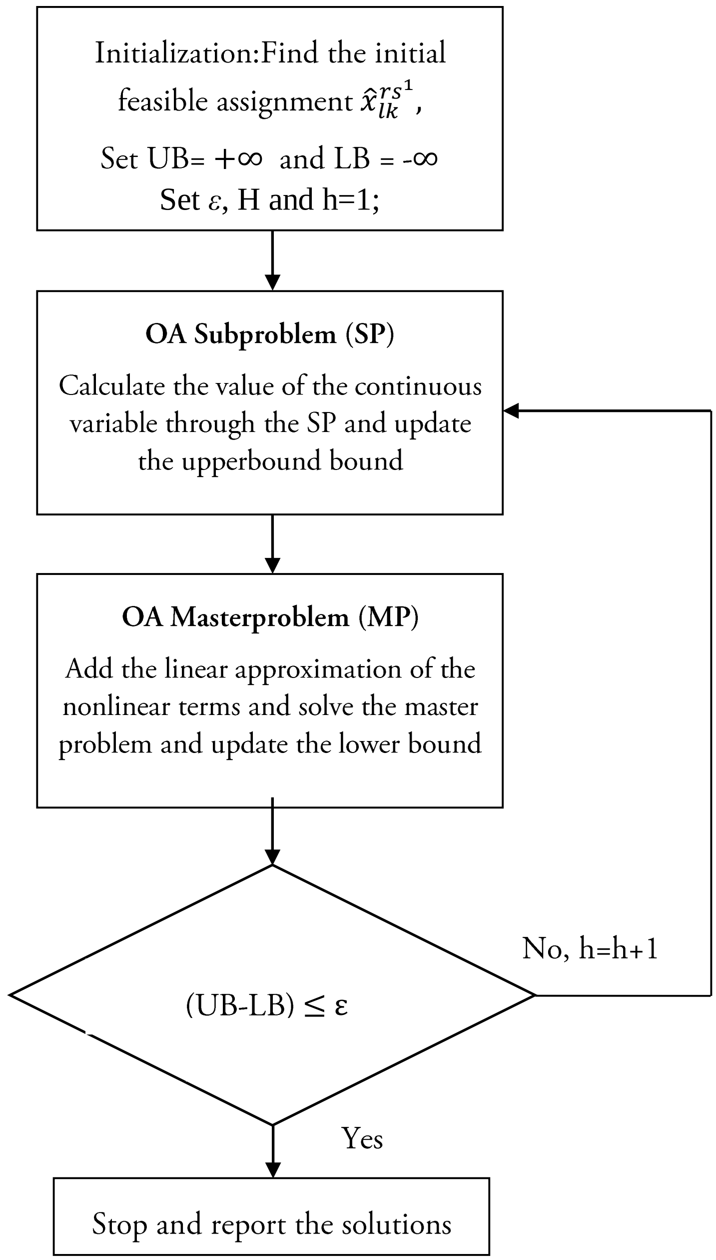


Figure 5-1: Outer Approximation (OA) framework

5.3 Numerical Experiment

In this section, a series of numerical experiments are designed to assess the performance of the proposed algorithm for solving the non-additive shortest path problem. The experiments have been conducted on six networks of varying sizes. The network examples are obtained from Bar-Gera website based on the best known link flow solutions. We assume different combinations of path travel cost functions to check the algorithm running time. The details on the size of the test networks and the travel cost function combinations are presented in Table 5-1 and Table 5-2. The OA algorithm for the non-additive shortest path problem is implemented in GAMS and we utilized CPLEX 12.4 to solve the MILP MP. All the experiments are tested on a Dell Studio PC with Intel Core i5 processor running at 3.3 GHz and 8 GB of RAM under a 64 bit windows operating system.

Table 5-1 Test Networks

Networks	Number of Links(A)	Number of Nodes (N)
Sioux Falls	76	24
Anaheim	914	416
Barcelona	2522	1020
Chicago Sketch	2250	933
Chicago Regional	39,018	12982
Philadelphia	40,003	13,389

5.3.1 Numerical Experiment 1

The aim of this experiment is to compare the efficiency of the proposed OA algorithm on different networks of varying sizes. In particular, we tested the algorithm on the networks presented in Table 5-1 based on the path cost functions defined in Table 5-2. The functions considered in Table 5-2 are all convex and differentiable by definition provided that the associated coefficients are all positive. The functions are selected in away to capture the changes in travel cost with corresponding decrease or increase in travel time. More specifically, the first five combinations are representing increasing behavior in travel cost with increase in travel time while the functions in the sixth scenario have decreasing behavior with increase in travel time. It should be noted that in non-additive shortest path with increasing behavior travel cost functions delay or late arrival would be penalized while in functions with decreasing behavior we are concerned with the paths which minimum of attributes such as early arrival or unreliability. Also combinations seven and eight are comprised of both an increasing and decreasing functions which are looking to minimize the tradeoff between attributes such as delay and early arrival. The coefficients are generated as a random number between 1 and 3. We report the minimum, maximum, and the average amount of running time as well as minimum and maximum numbers of numbers of iterations as the performance measure across ten randomly generated coefficients. Also, function $\frac{b}{(c-\alpha)}$ is defined in the domain where $c>\alpha$ and c has been randomly selected in way which is meaningful for each network. Node one is selected as the origin node and the highest numbered node as the destination node for each network. For example for Barcelona we considered node 1 as the origin and node 1020 as the destination node and solved the non-additive shortest path problem based on the constructed O-D pair. Both OA and OA MP are solved to optimality.

Table 5-2: Travel Cost Functions

Scenario	$U_1(\alpha)$	$U_2(\alpha)$
S1	$a(\frac{\alpha}{c}+b)^2$	$a(\frac{\alpha}{c}+b)^4$
S2	$a(\frac{\alpha}{c}+b)^2$	$be^{\alpha/c}$
S3	$a(\frac{\alpha}{c}+b)^4$	$be^{\alpha/c}$
S4	$a(\frac{\alpha}{c}+b)^4$	$b/(c-\alpha)$
S5	$be^{\alpha/c}$	$b/(c-\alpha)$
S6	$be^{-c\alpha}$	$b/(c+\alpha)$
S7	$a(\frac{\alpha}{c}+b)^4$	$be^{-c\alpha}$
S8	$b/(\alpha+c)$	$be^{\alpha/c}$

Table 5-3: Results of Sioux Falls and Anaheim

Network	Scenario	Running Time			Iterations	
		Minimum	Average	Maximum	Minimum	Maximum
Sioux Falls	S1	0.108	0.1387	0.187	2	2
	S2	0.125	0.1469	0.203	2	2
	S3	0.114	0.1321	0.211	2	2
	S4	0.125	0.1438	0.157	2	2
	S5	0.14	0.1592	0.172	2	2
	S6	0.109	0.128	0.165	2	2
	S7	0.14	0.1529	0.181	2	2
	S8	0.124	0.1403	0.156	2	2
Anaheim	S1	0.124	0.139	0.172	2	2
	S2	0.11	0.1547	0.233	2	2
	S3	0.11	0.1592	0.218	2	2
	S4	0.156	0.2197	0.264	2	2
	S5	0.109	0.1824	0.25	2	2
	S6	0.046	0.0997	0.14	2	2
	S7	0.125	0.1859	0.235	2	3
	S8	0.219	0.9411	1.7	2	3

Table 5-4: Results of Chicago Sketch and Barcelona

Network	Scenario	Running Time			Iterations	
		Minimum	Average	Maximum	Minimum	Maximum
Chicago Sketch	S1	0.202	0.2622	0.374	2	2
	S2	0.233	0.2872	0.327	2	2
	S3	0.312	1.6412	2.966	2	3
	S4	0.25	0.3013	0.391	2	4
	S5	0.25	0.2735	0.297	2	2
	S6	0.203	0.2368	0.265	2	2
	S7	0.202	0.2622	0.374	2	2
	S8	0.514	1.3539	5.974	2	4
Barcelona	S1	0.296	0.3279	0.36	2	2
	S2	0.265	0.2948	0.343	2	2
	S3	0.326	1.7746	3.196	2	3
	S4	0.297	0.334	0.422	2	2
	S5	0.264	0.3305	0.374	2	2
	S6	0.109	0.1513	0.218	2	2
	S7	0.437	0.6158	0.842	3	4
	S8	0.328	0.917273	3.522	2	4

Table 5-5: Results of Chicago and Philadelphia

Network	Scenario	Running Time			Iterations	
		Minimum	Average	Maximum	Minimum	Maximum
Chicago	S1	1.841	3.7504	4.415	2	2
	S2	3.588	3.7832	4.15	2	2
	S3	4.477	24.2762	44.084	2	4
	S4	4.01	4.2384	4.461	2	2
	S5	4.368	5.0607	6.241	2	3
	S6	1.699	1.7733	1.825	2	2
	S7	4.258	6.2364	7.94	2	3
	S8	9.314	15.9244	34.117	2	4
Philadelphia	S1	6.193	6.4083	6.692	2	2
	S2	7.987	11.8637	14.305	2	3
	S3	15.1	29.42556	35.677	2	4
	S4	15.132	24.5824	36.77	2	4
	S5	14.226	15.0932	16.49	2	2
	S6	0.654	1.6875	2.074	2	2
	S7	8.565	11.1542	14.29	3	3
	S8	14.008	17.3363	25.538	3	4

Tables 3-5 to 5-5 present the computational time for solving the non-additive shortest path problem for six varying size networks based on eight different combinations of path travel cost functions. The results show that the running time would grow with increase in network size which is an expected outcome. For example the average running time for the Sioux Falls case is in the first scenario is found to be 0.138 seconds while this increases to 6.408 seconds in the Philadelphia network. Moreover, we can observe from the achieved results that scenario three for path travel cost which is a combination of exponential and polynomial function is the hardest combination to solve as the computational time for this scenario tends to be more than the other combinations.

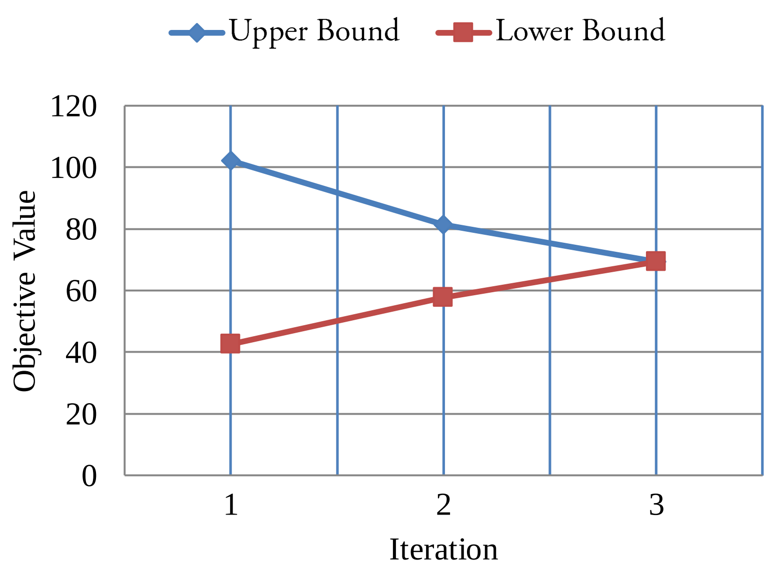


Figure 5-2: Convergence trajectory of OA algorithm for Philadelphia scenario 8

Figure 5-2 demonstrates the trajectory of the OA algorithm for Philadelphia network based on scenario 8 which shows how that the upper bound and lower bound solutions reached the same value after three iterations and the algorithm terminated.

5.3.2 Numerical Experiment 2

In this experiment, the performance of the OA algorithm was also tested for various O-D pairs of the test networks in this dissertation with travel cost functions based on scenarios 3, 6, and 8 of Table (5-2). A hundred O-D pairs were randomly selected for all the test networks except Sioux Falls for which we selected 24 O-D pairs. Similar to the previous experiment the coefficients for travel cost functions are generated randomly between zero and three. We solved the non-additive shortest path problem for every selected O-D pair and reported the computational effort and number of iterations. More specifically, minimum, average and maximum amount of computational time required to solve the shortest path for selected O-D pair along with number of iterations were reported to provide more insight about the algorithm performance. It should be noted that across all the test experiments in this part we solved OA MP within 0.5 percent of optimality gap whereas the OA tolerance was set at zero percent to assure the global optimality of the solution.

Table 5-6 Results of the second numerical experiment

Network	Scenario	Running Time			Iterations	
		Minimum	Average	Maximum	Minimum	Maximum
Philadelphia	S3	1.591	14.114	46.332	2	3
	S6	0.764	8.177	28.439	2	2
	S8	3.289	6.398	29.278	2	4
Chicago	S3	1.357	2.874	26.63	2	3
	S6	2.074	4.117	25.865	2	2
	S8	5.378	17.247	35.923	2	4
Chicago Sketch	S3	0.121	0.251	0.457	2	2
	S6	0.031	0.269	0.593	2	2
	S8	0.324	1.659	6.271	2	3
Barcelona	S3	0.113	0.301	0.783	2	2
	S6	0.123	0.209	0.794	2	2
	S8	0.215	1.714	4.963	2	3
Anaheim	S3	0.125	0.198	0.237	2	2
	S6	0.098	0.137	0.209	2	2
	S8	0.141	1.344	2.502	2	3
Sioux Falls	S3	0.087	0.156	0.243	1	2
	S6	0.065	0.167	0.251	1	2
	S8	0.098	0.1665	0.1954	1	2

The above Table provides the minimum, average, and maximum amount of computational time for the selected O-D pairs across all the networks. We can observe from the results the variation of computational effort is higher in the large networks such as Chicago and Philadelphia when compared to smaller networks for which the difference between the average, minimum, and the maximum seems to be negligible. This is mostly due to the fact that in large networks the variation of paths for an O-D pair in terms of number of underlying arcs is more than small networks such as Sioux Falls or Anaheim. Also this results show that in all the cases algorithm converges to the exact solution in maximum of four iterations which confirms the fact that the OA can provide the exact global solution in finite and relatively few numbers of iterations.

Chapter 6 Conclusion

6.1 Summary and Implications

Uncertainty is ubiquitous to network infrastructures. Typically: there are two major concerns ahead of the decision makers in coping with uncertainty: (i) How to model the prevailing uncertainty conditions to reflect the real world concerns (ii) How to solve the proposed model in a computationally tractable fashion?

This dissertation develops a general framework for the robust shortest path problem with different assumptions on the uncertainty conditions and also proposes an efficient algorithm which is capable of providing the exact global solution for different classes of non-additive shortest path problems. In particular, this dissertation studies two formulations for the robust shortest path problem based on assumptions of availability and unavailability of the information on the distribution of the links travel time. Towards this goal, two formulations are studied and solved for varying network instances. Finally, since both the proposed formulations falls into the category of non-additive shortest path problems with non-additive convex cost function, a general formulation for the convex non-additive shortest path problem has been studied and solved. The proposed modeling formulation is particularly attractive as it provides the global optimal solution for this class of programs. An overview of the dissertation contributions is provided in the next section followed by directions for future research.

6.2 Dissertation Contributions and Conclusions

The contributions of this dissertation are:

6.2.1 An efficient algorithm for mean-standard deviation shortest path problem

In the third chapter the general mean-standard deviation shortest path problem assuming the availability of the link travel cost correlation matrix has been studied. The contribution to the literature is the development of an efficient outer approximation algorithm for this problem, which was demonstrated to be substantially faster than standard algorithms. The algorithm's efficiency is due to a nonlinear subproblem which can be solved in closed form, and because it minimizes the impact of large correlation matrices. At every iteration only those elements of the covariance matrix which are constructing the shortest paths are included in the model and there is no need to account for all other elements of the network covariance matrix. This enables the OA algorithm to be highly efficient in handling large correlation matrices. The computational performance of the algorithm was tested on four standard transportation networks of increasing sizes. The algorithm was found to deliver the optimal solution for all test networks, for different values of variance and covariance structures. One of the main reasons for the efficiency of the OA is its ability to handle large correlation matrices. Notably, the outer approximation algorithm was found to significantly outperform the standard GAMS/CPLEX conic quadratic programming solver

6.2.2 Shortest Path Problem under Uncertain Link Travel Cost Distribution

The robust modeling formulation for the shortest path problem while assuming limited information on the specification of link travel costs function has been developed in chapter 5. In particular, the link travel costs are assumed to be described by the mean travel cost and an uncertainty term representing a combination of various uncertainty sources such as bad weather, incidents etc. Considering that the uncertainties belong to a set with ellipsoidal shape the robust formulation for the shortest path problem can be formulated as a MINLP which is then transformed to a MICQP program. The robust variant due to non-additive property of the cost function is complicated to solve for which a novel outer approximation algorithm promising global optimality within reasonable amount of time has been proposed. The performance of the formulation along with the suggested solution approach has been evaluated on six different small to large size test networks. The robust solutions obtained by the robust optimization approach mostly provide a different routing decision with slightly higher cost in compare to the deterministic setting in order to hedge against the realized uncertainty. Finally, through extensive numerical study the proposed solution framework is shown to be more efficient when compared to the existing MICQP solver such as CPLEX.

6.2.3 Non-additive Shortest Path Problem

The two formulations presented in chapters 4 and 5 falls into the category of the non-additive shortest path problems. Therefore, in chapter 6, a general formulation for the non-additive shortest path problem has been investigated. The contribution to the existing literature is the application of the OA for shortest path problem with non-additive path travel cost function. In particular, considering a shortest path problem with multiple attributes each of them have a continuous convex differentiable function. The objective of the non-additive shortest path is to find the best routing decision in presence of the nonlinear network user's preferences. Several numerical experiments were conducted to test the efficiency and correctness of the algorithm on real world networks with varying size. The numerical results indicated that the algorithm is capable of providing the exact global solution for large size networks.

6.3 Directions for Future Research

The work conducted in this dissertation can be extended in multiple ways. The extensions can be either along different modeling assumptions or enhancing the performance of the solution algorithm.

- An immediate extension can be the application of the current model in a traffic assignment framework since the shortest path can potentially serves as the subproblem to network assignment problems. All three variants of the shortest path studied in this dissertation can be incorporated within a traffic assignment problem to account for equilibrium behavior of network users under uncertainty.
- Study the difference between the optimal path obtained from the solution of mean-standard deviation shortest path problem and the robust version with unknown travel cost distribution function is an interesting extension to be considered in future.
- Since the OA algorithm solves the initial nonlinear formulation through a series of MILP one possible extension for improving the performance of the OA algorithm worthy of study can be the application of heuristics or network flow algorithm in intelligently solving the OA MP.

- The robust time dependent shortest path variant under uncertain travel cost using the minimax and not least expected travel time perspective is another avenue of research worth of study.
- Extending the methodology presented in chapter 6 for the link travel costs with non-convex, non-differentiable cases is another line of research which can generalize the methodology to shortest path with general non-additive cost function.

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